Handbook of Complex Variables

Steven G. Krantz

Handbook of Complex Variables

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To the memory of Lars Valerian Ahlfors, 1907–1996.

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Preface

This book is written to be a convenient reference for the working scientist, student, or engineer who needs to know and use basic concepts in complex analysis. It is not a book of mathematical theory. It is instead a book of mathematical practice.

All the basic ideas of complex analysis, as well as many typical applications, are treated. Since we are not developing theory and proofs, we have not been obliged to conform to a strict logical ordering of topics. Instead, topics have been organized for ease of reference, so that cognate topics appear in one place.

Required background for reading the text is minimal: a good grounding in (real variable) calculus will suffice. However, the reader who gets maximum utility from the book will be that reader who has had a course in complex analysis at some time in his life. This book is a handy compendium of all basic facts about complex variable theory. But it is not a textbook, and a person would be hard put to endeavor to learn the subject by reading this book.

Understanding that many readers are not primary users of mathematics, we have included an extensive glossary and table of notation. The glossary contains not only all the major terminology of complex variable theory but also background terminology such as "uniform convergence," "equivalence class," "compact set," and "accumulation point." We have also included a complete List of Notation, a Table of Laplace Transforms, and a Guide to the Literature. The latter is a list of some of the major works in complex analysis, grouped by type. The book concludes with a References section and a rather thorough Index.

We have made every effort to give thorough references for all topics. These references provide further reading, theory, proofs, and exercises.

The notation used in this book is quite standard in the mathematical world. But the reader should be aware that certain specialized fields have their own particular notation. For example, electrical engineering uses the letter j to denote the square root of -1, whereas mathematicians use the letter i for this purpose. We leave it to the reader to make whatever small adaptations are necessary to render the contents of this book consistent with his particular field; we would be doing the reader a disservice to introduce all possible notational variants.

In order to make the book as easy to use and as self-contained as possible, we have occasionally repeated topics or formulas. Surely it is more convenient for readers to have the necessary formula right on the page they

are reading than to be flipping back and forth among several disparate pages. Readers will also find the extensive cross-referencing to be useful.

Because we view the typical reader of this book as one who is interested in applications of complex functions, we have made a special effort to include material on areas of engineering and physics in which complex variable theory is applied. This discussion is gathered at the end of the book, in Chapters 14 and 15.

The book concludes (Chapter 16) with a brief discussion of software that is useful for understanding concepts from complex variable theory.

We hope that this book will be a valuable tool for the mathematical scientist.

It is a pleasure to thank the many friends and colleagues who made this book possible. Ken Rosen first piqued my interest in writing a handbook. My editor, Wayne Yuhasz, guided me through every step of the process of writing the book and provided technical support when necessary. Sean Davey and Clovis Tondo contributed their TEXpertise. Lynn Apfel gave me some useful ideas about charts and tables.

Special thanks go to Kitty Laird, who put in untold hours rendering figures with Corel Draw. I am grateful to George Kamberov and to Stanley Sawyer, who patiently taught me how to incorporate graphics into a TEX document. Finally, I thank Louise Farkas for her splendid copyediting. Without all these good people, this book could never have been completed in its present form.

Responsibility for all extant shortcomings in the text is mine alone. Suggestions for improvement, or identification of errata, are always appreciated.

St. Louis, Missouri — Steven G. Krantz

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Chapter 1

The Complex Plane

1.1 Complex Arithmetic

1.1.1 The Real Numbers

The real number system consists of both the rational numbers (numbers with terminating or repeating decimal expansions) and the irrational numbers (numbers with infinite, non-repeating decimal expansions). The real numbers are denoted by the symbol $\mathbb{R}$. We let $\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}\,,\ y \in \mathbb{R}\}$ (Figure 1.1).

1.1.2 The Complex Numbers

The complex numbers $\mathbb{C}$ consist of $\mathbb{R}^2$ equipped with some special algebraic operations. One defines

$$(x, y) + (x', y') = (x + x', y + y')\,,$$
$$(x, y) \cdot (x', y') = (xx' - yy', xy' + yx').$$

These operations of $+$ and $\cdot$ are commutative and associative.

We denote $(1, 0)$ by 1 and $(0,1)$ by i. If $\alpha \in \mathbb{R}$, then we identify α with the complex number $(\alpha, 0)$. Using this notation, we see that

$$\alpha \cdot (x, y) = (\alpha, 0) \cdot (x, y) = (\alpha x, \alpha y). \tag{1.1.2.1}$$

Thus every complex number (x, y) can be written in one and only one fashion in the form $x \cdot 1 + y \cdot i$ with $x, y \in \mathbb{R}$. We usually write the number even more succinctly as $x + iy$. The laws of addition and multiplication become

$$(x + iy) + (x' + iy') = (x + x') + i(y + y'),$$
$$(x + iy) \cdot (x' + iy') = (xx' - yy') + i(xy' + yx').$$

S. G. Krantz, *Handbook of Complex Variables*
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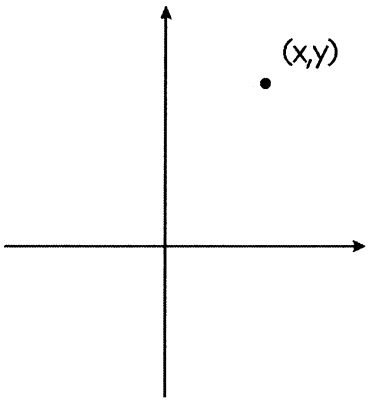

FIGURE 1.1
A point in the plane.

Observe that $i \cdot i = -1$. Finally, the multiplication law is consistent with the scalar multiplication introduced in line (1.1.2.1).

The symbols z, w, ζ are frequently used to denote complex numbers. We usually take $z = x + iy$, $w = u + iv$, $\zeta = \xi + i\eta$. The real number x is called the *real part* of z and is written $x = \operatorname{Re} z$. The real number y is called the *imaginary part* of z and is written $y = \operatorname{Im} z$.

The complex number $x - iy$ is by definition the complex *conjugate* of the complex number $x + iy$. If $z = x + iy$, then we denote the conjugate of z with the symbol $\overline{z}$; thus $\overline{z} = x - iy$.

1.1.3 Complex Conjugate

Note that $z + \overline{z} = 2x$, $z - \overline{z} = 2iy$. Also

$$\overline{z + w} = \overline{z} + \overline{w}, \tag{1.1.3.1}$$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}. \tag{1.1.3.2}$$

A complex number is real (has no imaginary part) if and only if $z = \overline{z}$. It is imaginary (has no real part) if and only if $z = -\overline{z}$.

1.1.4 Modulus of a Complex Number

The ordinary Euclidean distance of (x, y) to $(0, 0)$ is $\sqrt{x^2 + y^2}$ (Figure 1.2). We also call this number the *modulus* of the complex number $z = x+iy$ and we write $|z| = \sqrt{x^2 + y^2}$. Note that

$$z \cdot \overline{z} = x^2 + y^2 = |z|^2.$$

The distance from z to w is $|z-w|$. We also have the formulas $|z \cdot w| = |z| \cdot |w|$ and $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.

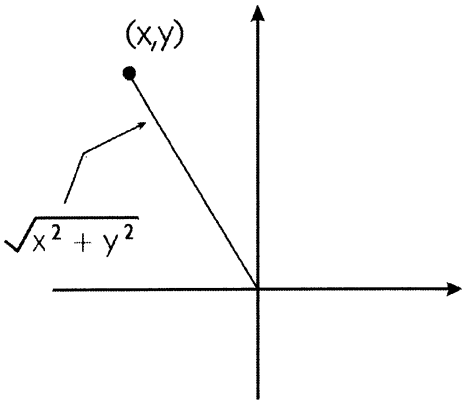

FIGURE 1.2
Distance to the origin or modulus.

1.1.5 The Topology of the Complex Plane

If P is a complex number and $r > 0$, then we set

$$D(P,r) = \{z \in \mathbb{C} : |z - P| < r\} \tag{1.1.5.1}$$

and

$$\overline{D}(P,r) = \{z \in \mathbb{C} : |z - P| \leq r\}. \tag{1.1.5.2}$$

The first of these is the *open disc with center P and radius r*; the second is the *closed disc with center P and radius r* (Figure 1.3). We often use the simpler symbols D and $\overline{D}$ to denote, respectively, the discs $D(0,1)$ and $\overline{D}(0,1)$.

We say that a set $U \subseteq \mathbb{C}$ is *open* if, for each $P \in \mathbb{C}$, there is an $r > 0$ such that $D(P,r) \subseteq U$. Thus an open set is one with the property that each point P of the set is surrounded by neighboring points that are still in the set (that is, the points of distance less than r from P)—see Figure 1.4. Of course the number r will depend on P. As examples, $U = \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$ is open, but $F = \{z \in \mathbb{C} : \operatorname{Re} z \leq 1\}$ is not (Figure 1.5).

A set $E \subseteq \mathbb{C}$ is said to be *closed* if $\mathbb{C} \setminus E \equiv \{z \in \mathbb{C} : z \notin E\}$ (the complement of E in $\mathbb{C}$) is open. The set F in the last paragraph is closed.

It is *not* the case that any given set is either open or closed. For example, the set $W = \{z \in \mathbb{C} : 1 < \operatorname{Re} z \leq 2\}$ is *neither* open *nor* closed (Figure 1.6).

We say that a set $E \subset \mathbb{C}$ is *connected* if there do not exist non-empty disjoint open sets U and V such that $E = (U \cap E) \cup (V \cap E)$. Refer to Figure 1.7 for these ideas. It is a useful fact that if E is an open set, then E is connected if and only if it is path-connected; this means that any two points of E can be connected by a continuous path or curve. See Figure 1.8.

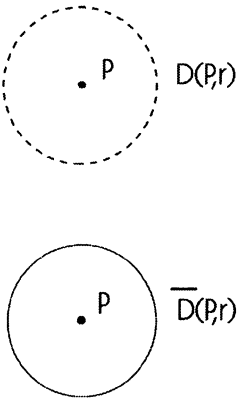

FIGURE 1.3
An open disc and a closed disc.

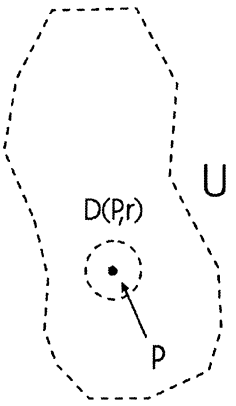

FIGURE 1.4
An open set.

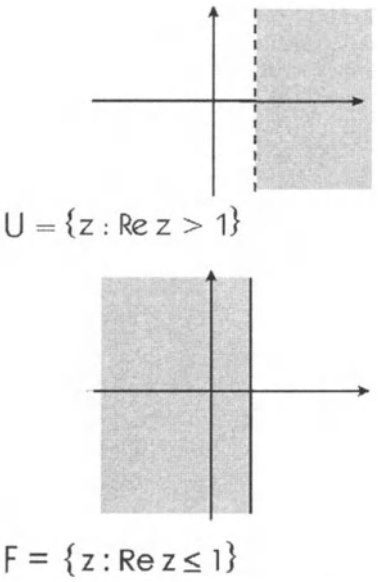

FIGURE 1.5
An open set and a non-open set.

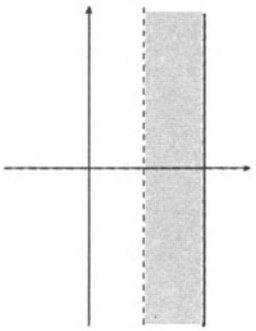

FIGURE 1.6
A set that is neither open nor closed.

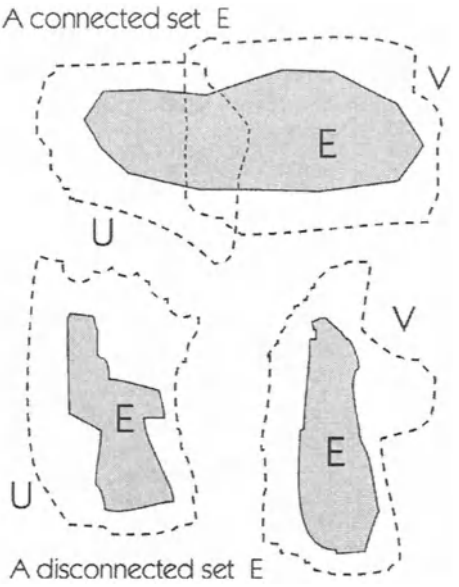

FIGURE 1.7
A connected set and a disconnected set.

1.1.6 The Complex Numbers as a Field

Let 0 denote the number $0 + i0$. If $z \in \mathbb{C}$, then $z + 0 = z$. Also, letting $-z = -x - iy$, we have $z + (-z) = 0$. So every complex number has an additive inverse, and that inverse is unique.

Since $1 = 1 + i0$, it follows that $1 \cdot z = z \cdot 1 = z$ for every complex number z. If $z \neq 0$, then $|z|^2 \neq 0$ and

$$z \cdot \left(\frac{\overline{z}}{|z|^2} \right) = \frac{|z|^2}{|z|^2} = 1. \tag{1.1.6.1}$$

So every non-zero complex number has a multiplicative inverse, and that inverse is unique. It is natural to define $1/z$ to be the multiplicative inverse $\overline{z}/|z|^2$ of z and, more generally, to define

$$\frac{z}{w} = z \cdot \frac{1}{w} = \frac{z\overline{w}}{|w|^2} \qquad \text{for } w \neq 0. \tag{1.1.6.2}$$

We also have $\overline{z/w} = \overline{z}/\overline{w}$.

Multiplication and addition satisfy the usual distributive, associative, and commutative laws. Therefore $\mathbb{C}$ is a *field* (see [HER]). The field $\mathbb{C}$ contains a copy of the real numbers in an obvious way:

$$\mathbb{R} \ni x \mapsto x + i0 \in \mathbb{C}. \tag{1.1.6.3}$$

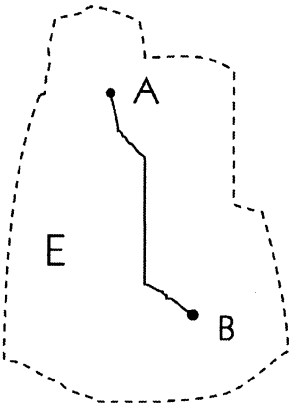

FIGURE 1.8
An open set is connected if and only if it is path-connected.

This identification respects addition and multiplication. So we can think of $\mathbb{C}$ as a field extension of $\mathbb{R}$: it is a larger field which contains the field $\mathbb{R}$.

1.1.7 The Fundamental Theorem of Algebra

It is not true that every non-constant polynomial with real coefficients has a real root. For instance, $p(x) = x^2 + 1$ has no real roots. The Fundamental Theorem of Algebra states that every polynomial with complex coefficients has a complex root (see the treatment in §§3.1.4 below). The complex field $\mathbb{C}$ is the *smallest* field that contains $\mathbb{R}$ and has this so-called algebraic closure property.

1.2 The Exponential and Applications

1.2.1 The Exponential Function

We define the complex exponential as follows:

(1.2.1.1) If $z = x$ is real, then

$$e^z = e^x \equiv \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

as in calculus. Here ! denotes "factorial": $j! = j \cdot (j-1) \cdot (j-2) \cdots 3 \cdot 2 \cdot 1$.

(1.2.1.2) If $z = iy$ is pure imaginary, then

$$e^z = e^{iy} \equiv \cos y + i \sin y.$$

(1.2.1.3) If $z = x + iy$, then

$$e^z = e^{x+iy} \equiv e^x \cdot e^{iy} = e^x \cdot (\cos y + i \sin y).$$

Part and parcel of the last definition of the exponential is the following complex-analytic definition of the sine and cosine functions:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \tag{1.2.1.4}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{1.2.1.5}$$

Note that when $z = x + i0$ is real this new definition coincides with the familiar Euler formula from calculus:

$$e^{ix} = \cos x + i \sin x. \tag{1.2.1.6}$$

1.2.2 The Exponential Using Power Series

It is also possible to define the exponential using power series:

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}. \tag{1.2.2.1}$$

Either definition (that in §§1.2.1 or in §§1.2.2) is correct for any z, and they are logically equivalent.

1.2.3 Laws of Exponentiation

The complex exponential satisfies familiar rules of exponentiation:

$$e^{z+w} = e^z \cdot e^w \quad \text{and} \quad (e^z)^w = e^{zw}. \tag{1.2.3.1}$$

Also

$$\left(e^z\right)^n = \underbrace{e^z \cdots e^z}_{n \text{ times}} = e^{nz}. \tag{1.2.3.2}$$

1.2.4 Polar Form of a Complex Number

A consequence of our first definition of the complex exponential —see (1.2.1.2)—is that if $\zeta \in \mathbb{C}$, $|\zeta| = 1$, then there is a unique number θ,

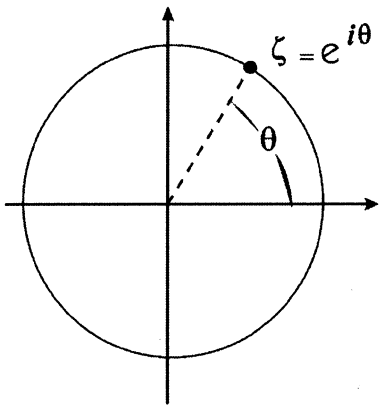

FIGURE 1.9
Polar coordinates of a point in the plane.

$0 \leq \theta < 2\pi$, such that $\zeta = e^{i\theta}$ (see Figure 1.9). Here θ is the (signed) angle between the positive x axis and the ray $\overrightarrow{0\zeta}$.

Now if z is any non-zero complex number, then

$$z = |z| \cdot \left(\frac{z}{|z|}\right) \equiv |z| \cdot \zeta \tag{1.2.4.1}$$

where $\zeta = z/|z|$ has modulus 1. Again letting θ be the angle between the real axis and $\overrightarrow{0\zeta}$, we see that

$$\begin{aligned} z &= |z| \cdot \zeta \\ &= |z| e^{i\theta} \\ &= re^{i\theta}\,, \end{aligned} \tag{1.2.4.2}$$

where $r = |z|$. This form is called the *polar* representation for the complex number z. (Note that some classical books write the expression $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ as $z = r\mathrm{cis}\,\theta$. The reader should be aware of this notation, though we shall not use it in this book.)

EXAMPLE 1.2.4.1 Let $z = 1 + \sqrt{3}i$. Then $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$. Hence

$$z = 2 \cdot \left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right).$$

The number in parenthesis subtends an angle of $\pi/3$ with the positive x-axis. Therefore

$$1 + \sqrt{3}i = z = 2 \cdot e^{i\pi/3}. \qquad \square$$

It is often convenient to allow angles that are greater than or equal to 2π in the polar representation; when we do so, the polar representation is no longer unique. For if k is an integer, then

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta = \cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi) \\ &= e^{i(\theta+2k\pi)}. \end{aligned} \tag{1.2.4.3}$$

1.2.5 Roots of Complex Numbers

The properties of the exponential operation can be used to find the n^{th} roots of a complex number.

EXAMPLE 1.2.5.1 To find all sixth roots of 2, we let $re^{i\theta}$ be an arbitrary sixth root of 2 and solve for r and θ. If

$$\left(re^{i\theta}\right)^6 = 2 = 2 \cdot e^{i0} \tag{1.2.5.1.1}$$

or

$$r^6 e^{i6\theta} = 2 \cdot e^{i0}, \tag{1.2.5.1.2}$$

then it follows that $r = 2^{1/6} \in \mathbb{R}$ and $\theta = 0$ solve this equation. So the real number $2^{1/6} \cdot e^{i0} = 2^{1/6}$ is a sixth root of two. This is not terribly surprising, but we are not finished.

We may also solve

$$r^6 e^{i6\theta} = 2 = 2 \cdot e^{2\pi i}. \tag{1.2.5.1.3}$$

Hence

$$r = 2^{1/6}\ ,\ \theta = 2\pi/6 = \pi/3. \tag{1.2.5.1.4}$$

This gives us the number

$$2^{1/6}e^{i\pi/3} = 2^{1/6}\left(\cos\pi/3 + i\sin\pi/3\right) \qquad = 2^{1/6}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \tag{1.2.5.1.5}$$

as a sixth root of two. Similarly, we can solve

$$\begin{aligned} r^6 e^{i6\theta} &= 2 \cdot e^{4\pi i} \\ r^6 e^{i6\theta} &= 2 \cdot e^{6\pi i} \\ r^6 e^{i6\theta} &= 2 \cdot e^{8\pi i} \\ r^6 e^{i6\theta} &= 2 \cdot e^{10\pi i} \end{aligned}$$

to obtain the other four sixth roots of 2:

$$2^{1/6}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \tag{1.2.5.1.6}$$

$$-2^{1/6} \tag{1.2.5.1.7}$$

$$2^{1/6}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) \tag{1.2.5.1.8}$$

$$2^{1/6}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right). \tag{1.2.5.1.9}$$

These are in fact all the sixth roots of 2. □

EXAMPLE 1.2.5.2 Let us find all third roots of i. We begin by writing i as

$$i = e^{i\pi/2}. \tag{1.2.5.2.1}$$

Solving the equation

$$(re^{i\theta})^3 = i = e^{i\pi/2} \tag{1.2.5.2.2}$$

then yields $r = 1$ and $\theta = \pi/6$.

Next, we write $i = e^{i5\pi/2}$ and solve

$$(re^{i\theta})^3 = e^{i5\pi/2} \tag{1.2.5.2.3}$$

to obtain that $r = 1$ and $\theta = 5\pi/6$.

Lastly, we write $i = e^{i9\pi/2}$ and solve

$$(re^{i\theta})^3 = e^{i9\pi/2} \tag{1.2.5.2.4}$$

to obtain that $r = 1$ and $\theta = 9\pi/6 = 3\pi/2$.

In summary, the three cube roots of i are

$$e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}, \tag{1.2.5.2.5}$$

$$e^{i5\pi/6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}, \tag{1.2.5.2.6}$$

$$e^{i3\pi/2} = -i. \tag{1.2.5.2.7}$$

□

1.2.6 The Argument of a Complex Number

The (non-unique) angle θ associated to a complex number $z \neq 0$ is called its *argument*, and is written $\arg z$. For instance, $\arg(1+i) = \pi/4$. But it is also correct to write $\arg(1+i) = 9\pi/4, 17\pi/4, -7\pi/4$, etc. We generally choose the argument θ to satisfy $0 \leq \theta < 2\pi$. This is the principal branch of the argument—see §§10.1.2, §§10.4.2.

Under multiplication of complex numbers, arguments are additive and moduli multiply. That is, if $z = re^{i\theta}$ and $w = se^{i\psi}$ then

$$z \cdot w = re^{i\theta} \cdot se^{i\psi} = (rs) \cdot e^{i(\theta+\psi)}. \tag{1.2.6.1}$$

1.2.7 Fundamental Inequalities

We next record a few inequalities.

The Triangle Inequality: If $z, w \in \mathbb{C}$, then

$$|z + w| \leq |z| + |w|. \tag{1.2.7.1}$$

More generally,

$$\left| \sum_{j=1}^{n} z_j \right| \leq \sum_{j=1}^{n} |z_j|. \tag{1.2.7.2}$$

The Cauchy-Schwarz Inequality: If $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ are complex numbers, then

$$\left| \sum_{j=1}^{n} z_j w_j \right|^2 \leq \left[\sum_{j=1}^{n} |z_j|^2 \right] \cdot \left[\sum_{j=1}^{n} |w_j|^2 \right]. \tag{1.2.7.3}$$

1.3 Holomorphic Functions

1.3.1 Continuously Differentiable and C^k Functions

In this book we will frequently refer to a *domain* or a *region* $U \subseteq \mathbb{C}$. Usually this will mean that U is an open set and that U is connected (see §1.1.5).

Holomorphic functions are a generalization of complex polynomials. But they are more flexible objects than polynomials. The collection of all polynomials is closed under addition and multiplication. However, the collection of all holomorphic functions is closed under reciprocals, inverses, exponentiation, logarithms, square roots, and many other operations as well.

There are several different ways to introduce the concept of holomorphic function. They can be defined by way of power series, or using the complex derivative, or using partial differential equations. We shall touch on all these approaches; but our initial definition will be by way of partial differential equations.

If $U \subseteq \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is a continuous function, then f is called C^1 (or *continuously differentiable*) on U if $\partial f/\partial x$ and $\partial f/\partial y$ exist and are *continuous* on U. We write $f \in C^1(U)$ for short.

More generally, if $k \in \{0, 1, 2, ...\}$, then a real-valued function f on U is called C^k (k times continuously differentiable) if all partial derivatives of f up to and including order k exist and are continuous on U. We write in this case $f \in C^k(U)$. In particular, a C^0 function is just a continuous function.

A function $f = u + iv : U \to \mathbb{C}$ is called C^k if both u and v are C^k.

1.3.2 The Cauchy-Riemann Equations

If f is *any* complex-valued function, then we may write $f = u+iv$, where u and v are real-valued functions.

EXAMPLE 1.3.2.1 Consider

$$f(z) = z^2 = (x^2 - y^2) + i(2xy);$$

in this example $u = x^2 - y^2$ and $v = 2xy$. We refer to u as the *real part* of f and denote it by $\mathrm{Re}\, f$; we refer to v as the *imaginary part* of f and denote it by $\mathrm{Im}\, f$. □

Now we formulate the notion of "holomorphic function" in terms of the real and imaginary parts of f :

Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a C^1 function. Write

$$f(z) = u(x, y) + iv(x, y),$$

with $z = x + iy$ and u and v real-valued functions. If u and v satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.3.2.2}$$

at every point of U, then the function f is holomorphic (see §§1.3.4, where a formal definition of "holomorphic" is provided). The first order, linear partial differential equations in (1.3.2.2) are called the *Cauchy-Riemann equations.* A practical method for checking whether a given function is holomorphic is to check whether it satisfies the Cauchy-Riemann equations. Another practical method is to check that the function can be expressed in terms of z alone, with no $\overline{z}$'s present (see §§1.3.3).

1.3.3 Derivatives

We define, for $f = u + iv : U \to \mathbb{C}$ a C^1 function,

$$\frac{\partial}{\partial z} f \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1.3.3.1}$$

and

$$\frac{\partial}{\partial \overline{z}} f \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{1.3.3.2}$$

If $z = x + iy$, $\overline{z} = x - iy$, then one can check directly that

$$\frac{\partial}{\partial z} z = 1\,, \qquad \frac{\partial}{\partial z}\overline{z} = 0\,,$$

$$\frac{\partial}{\partial \overline{z}} z = 0\,, \qquad \frac{\partial}{\partial \overline{z}}\overline{z} = 1. \tag{1.3.3.3}$$

If a C^1 function f satisfies $\partial f/\partial z \equiv 0$ on an open set U, then f does not depend on z (but it *does* depend on $\overline{z}$). If instead f satisfies $\partial f/\partial \overline{z} \equiv 0$ on an open set U, then f does not depend on $\overline{z}$ (but it *does* depend on z). The condition $\partial f/\partial \overline{z} = 0$ is a reformulation of the Cauchy-Riemann equations—see §§1.3.4.

1.3.4 Definition of Holomorphic Function

Functions f that satisfy $(\partial/\partial\overline{z})f \equiv 0$ are the main concern of complex analysis. A continuously differentiable (C^1) function $f : U \to \mathbb{C}$ defined on an open subset U of $\mathbb{C}$ is said to be *holomorphic* if

$$\frac{\partial f}{\partial \overline{z}} = 0 \tag{1.3.4.1}$$

at every point of U. Note that this last equation is just a reformulation of the Cauchy-Riemann equations (§§1.3.2). To see this, we calculate:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \overline{z}} f(z) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)[u(z) + iv(z)] \\ &= \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + i\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]. \end{aligned} \tag{1.3.4.2}$$

Of course the far right-hand side cannot be identically zero unless each of its real and imaginary parts is identically zero. It follows that

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

These are the Cauchy-Riemann equations (1.3.2.2).

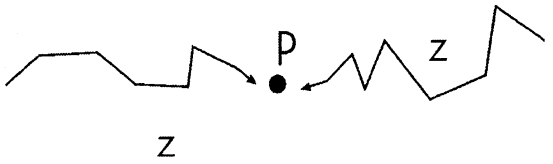

FIGURE 1.10
The point may approach P arbitrarily.

1.3.5 The Complex Derivative

Let $U \subseteq \mathbb{C}$ be open, $P \in U$, and $g : U \setminus \{P\} \to \mathbb{C}$ a function. We say that

$$\lim_{z \to P} g(z) = \ell \,, \quad \ell \in \mathbb{C} \,, \tag{1.3.5.1}$$

if for any $\epsilon > 0$ there is a $\delta > 0$ such that when $z \in U$ and $0 < |z - P| < \delta$ then $|g(z) - \ell| < \epsilon$.

We say that f possesses the *complex derivative* at P if

$$\lim_{z \to P} \frac{f(z) - f(P)}{z - P} \tag{1.3.5.2}$$

exists. In that case we denote the limit by $f'(P)$ or sometimes by

$$\frac{df}{dz}(P) \qquad \text{or} \qquad \frac{\partial f}{\partial z}(P). \tag{1.3.5.3}$$

This notation is consistent with that introduced in §§1.3.3: for a *holomorphic function*, the complex derivative calculated according to formula (1.3.5.2) or according to formula (1.3.3.1) is just the same. We shall say more about the complex derivative in §2.2.3 and §2.2.4.

It should be noted that, in calculating the limit in (1.3.5.2), z must be allowed to approach P from *any* direction (see Figure 1.10). As an example, the function $g(x, y) = x - iy$—equivalently, $g(z) = \overline{z}$—does *not* possess the complex derivative at 0. To see this, calculate the limit

$$\lim_{z \to P} \frac{g(z) - g(P)}{z - P}$$

with z approaching $P = 0$ through values $z = x + i0$. The answer is

$$\lim_{x \to 0} \frac{x - 0}{x - 0} = 1.$$

If instead z is allowed to approach $P = 0$ through values $z = iy$, then the value is

$$\lim_{y \to 0} \frac{-iy - 0}{iy - 0} = -1.$$

Observe that the two answers do not agree. *In order for the complex derivative to exist, the limit must exist and assume only one value no matter how z approaches P.* Therefore, this example g does not possess the complex derivative at $P = 0$. In fact a similar calculation shows that this function g does not possess the complex derivative at any point.

If a function f possesses the complex derivative at every point of its open domain U, then f is holomorphic. This definition is equivalent to definitions given in §§1.3.2, §§1.3.4. We repeat some of these ideas in §2.2.

1.3.6 Alternative Terminology for Holomorphic Functions

Some books use the word "analytic" instead of "holomorphic." Still others say "differentiable" or "complex differentiable" instead of "holomorphic." The use of the term "analytic" derives from the fact that a holomorphic function has a local power series expansion about each point of its domain (see §§3.1.6). In fact this power series property is a complete characterization of holomorphic functions; we shall discuss it in detail below. The use of "differentiable" derives from properties related to the complex derivative. These pieces of terminology and their significance will all be sorted out as the book develops. Somewhat archaic terminology for holomorphic functions, which may be found in older texts, are "regular" and "monogenic."

Another piece of terminology that is applied to holomorphic functions is "conformal" or "conformal mapping." "Conformality" is an important geometric property of holomorphic functions that make these functions useful for modeling incompressible fluid flow (§§14.2.2) and other physical phenomena. We shall discuss conformality in §§2.2.5. We shall treat physical applications of conformality in Chapter 14.

1.4 The Relationship of Holomorphic and Harmonic Functions

1.4.1 Harmonic Functions

A C^2 function u is said to be *harmonic* if it satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0. \tag{1.4.1.1}$$

This equation is called *Laplace's equation*, and is frequently abbreviated as

$$\triangle u = 0. \tag{1.4.1.2}$$

1.4.2 Holomorphic and Harmonic Functions

If f is a holomorphic function and $f = u + iv$ is the expression of f in terms of its real and imaginary parts, then both u and v are harmonic. A converse is true provided the functions involved are defined on a domain with no holes:

> If $\mathcal{R}$ is an open rectangle (or open disc) and if u is a real-valued harmonic function on $\mathcal{R}$, then there is a holomorphic function F on $\mathcal{R}$ such that $\operatorname{Re} F = u$. In other words, for such a function u there exists a harmonic function v defined on $\mathcal{R}$ such that $f \equiv u + iv$ is holomorphic on $\mathcal{R}$. Any two such functions v must differ by a real constant.
>
> More generally, if U is a region with no holes (a *simply connected* region—see §§2.3.3), and if u is harmonic on U, then there is a holomorphic function F on U with $\operatorname{Re} F = u$. In other words, for such a function u there exists a harmonic function v defined on U such that $f \equiv u + iv$ is holomorphic on U. Any two such functions v must differ by a constant. We call the function v a *harmonic conjugate* for u.

The displayed statement is false on a domain with a hole, such as an annulus. For example, the harmonic function $u = \log(x^2 + y^2)$, defined on the annulus $U = \{z : 1 < |z| < 2\}$, has no harmonic conjugate on U. See also §§7.1.4.

Chapter 2

Complex Line Integrals

2.1 Real and Complex Line Integrals

In this section we shall recast the line integral from calculus in complex notation. The result will be the complex line integral.

2.1.1 Curves

It is convenient to think of a *curve* as a (continuous) function γ from a closed interval $[a, b] \subseteq \mathbb{R}$ into $\mathbb{R}^2 \approx \mathbb{C}$. In practice it is useful *not* to distinguish between the *function* γ and the image (or set of points that make up the curve) given by $\{\gamma(t) : t \in [a, b]\}$. Refer to Figure 2.1.

It is often convenient to write

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) \qquad \text{or} \qquad \gamma(t) = \gamma_1(t) + i\gamma_2(t). \tag{2.1.1.1}$$

For example, $\gamma(t) = (\cos t, \sin t) = \cos t + i \sin t$, $t \in [0, 2\pi]$, describes the unit circle in the plane. The circle is traversed in a counterclockwise manner as t increases from 0 to 2π. See Figure 2.1.

2.1.2 Closed Curves

The curve $\gamma : [a, b] \to \mathbb{C}$ is called *closed* if $\gamma(a) = \gamma(b)$. It is called *simple, closed* (or Jordan) if the restriction of γ to the interval $[a, b)$ (which is commonly written $\gamma\big|_{[a,b)}$) is one-to-one *and* $\gamma(a) = \gamma(b)$ (Figures 2.1, 2.2). Intuitively, a simple, closed curve is a curve with no self-intersections, except of course for the closing up at $t = a, b$.

In order to work effectively with γ we need to impose on it some differentiability properties.

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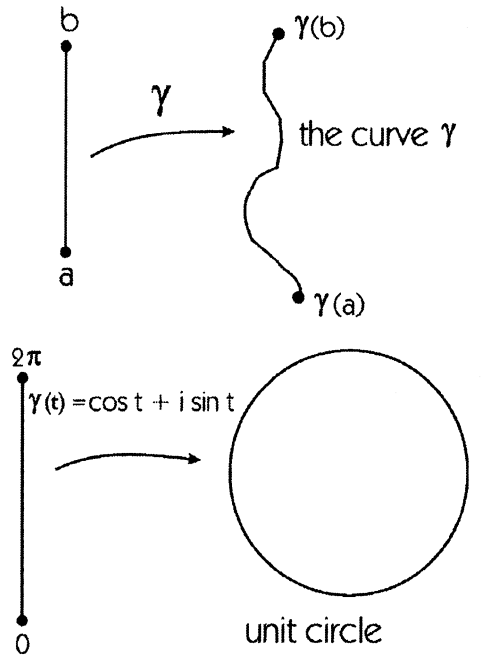

FIGURE 2.1
Two curves in the plane, one closed.

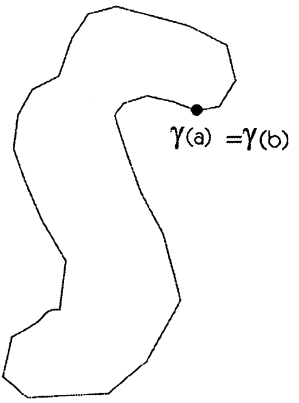

FIGURE 2.2
A simple, closed curve.

2.1.3 Differentiable and C^k Curves

A function $\varphi : [a,b] \to \mathbb{R}$ is called *continuously differentiable* (or C^1), and we write $\varphi \in C^1([a,b])$, if

(2.1.3.1) φ is continuous on $[a,b]$;
(2.1.3.2) φ' exists on (a,b);
(2.1.3.3) φ' has a continuous extension to $[a,b]$.

In other words, we require that

$$\lim_{t \to a^+} \varphi'(t) \quad \text{and} \quad \lim_{t \to b^-} \varphi'(t)$$

both exist.

Note that

$$\varphi(b) - \varphi(a) = \int_a^b \varphi'(t)\, dt, \tag{2.1.3.4}$$

so that the Fundamental Theorem of Calculus holds for $\varphi \in C^1([a,b])$.

A curve $\gamma : [a,b] \to \mathbb{C}$, with $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ is said to be *continuous* on $[a,b]$ if both γ_1 and γ_2 are. The curve is *continuously differentiable* (or C^1) on $[a,b]$, and we write

$$\gamma \in C^1([a,b]), \tag{2.1.3.5}$$

if γ_1, γ_2 are continuously differentiable on $[a,b]$. Under these circumstances we will write

$$\frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}. \tag{2.1.3.6}$$

We also write $\gamma'(t)$ for $d\gamma/dt$.

2.1.4 Integrals on Curves

Let $\psi : [a,b] \to \mathbb{C}$ be continuous on $[a,b]$. Write $\psi(t) = \psi_1(t) + i\psi_2(t)$. Then we define

$$\int_a^b \psi(t)\, dt \equiv \int_a^b \psi_1(t)\, dt + i \int_a^b \psi_2(t)\, dt. \tag{2.1.4.1}$$

We summarize the ideas presented thus far by noting that if $\gamma \in C^1([a,b])$ is complex-valued, then

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\, dt. \tag{2.1.4.2}$$

2.1.5 The Fundamental Theorem of Calculus along Curves

Now we state the Fundamental Theorem of Calculus (see [THO]) along curves.

Let $U \subseteq \mathbb{C}$ be a domain and let $\gamma : [a,b] \to U$ be a C^1 curve. If $f \in C^1(U)$, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \left(\frac{\partial f}{\partial x}(\gamma(t)) \cdot \frac{d\gamma_1}{dt} + \frac{\partial f}{\partial y}(\gamma(t)) \cdot \frac{d\gamma_2}{dt} \right) dt. \quad (2.1.5.1)$$

2.1.6 The Complex Line Integral

When f is holomorphic, then formula (2.1.5.1) may be rewritten (using the Cauchy-Riemann equations) as

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial f}{\partial z}(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)\, dt, \quad (2.1.6.1)$$

where, as earlier, we have taken $d\gamma/dt$ to be $d\gamma_1/dt + i d\gamma_2/dt$.

This latter result plays much the same role for holomorphic functions as does the Fundamental Theorem of Calculus for functions from $\mathbb{R}$ to $\mathbb{R}$. The expression on the right of (2.1.6.1) is called the *complex line integral* and is denoted

$$\oint_\gamma \frac{\partial f}{\partial z}(z)\, dz\,. \quad (2.1.6.2)$$

More generally, if g is *any* continuous function whose domain contains the curve γ, then the complex line integral of g along γ is defined to be

$$\oint_\gamma g(z)\, dz \equiv \int_a^b g(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)\, dt. \quad (2.1.6.3)$$

The whole concept of complex line integral is central to our further considerations in later sections. We shall use integrals like the one on the right of (2.1.6.1) even when f is not holomorphic; but we can be sure that the equality (2.1.6.1) holds only when f *is* holomorphic.

2.1.7 Properties of Integrals

We conclude this section with some easy but useful facts about integrals.

(2.1.7.1) If $\varphi : [a,b] \to \mathbb{C}$ is continuous, then

$$\left| \int_a^b \varphi(t)\, dt \right| \leq \int_a^b |\varphi(t)|\, dt. \quad (2.1.7.1.1)$$

(2.1.7.2) If $\gamma : [a, b] \to \mathbb{C}$ is a C^1 curve and φ is a continuous function on the curve γ, then

$$\left| \oint_\gamma \varphi(z)\, dz \right| \leq \left[\max_{t \in [a,b]} |\varphi(t)| \right] \cdot \ell(\gamma),$$

where

$$\ell(\gamma) \equiv \int_a^b |\varphi'(t)|\, dt$$

is the *length* of γ.

(2.1.7.3) The calculation of a complex line integral is independent of the way in which we parametrize the path:

> Let $U \subseteq \mathbb{C}$ be an open set and $F : U \to \mathbb{C}$ a continuous function. Let $\gamma : [a, b] \to U$ be a C^1 curve. Suppose that $\varphi : [c, d] \to [a, b]$ is a one-to-one, onto, increasing C^1 function with a C^1 inverse. Let $\widetilde{\gamma} = \gamma \circ \varphi$. Then
>
> $$\oint_{\widetilde{\gamma}} f\, dz = \oint_\gamma f\, dz. \qquad (2.1.7.3.1)$$
>
> This last statement implies that one can use the idea of the integral of a function f along a curve γ when the curve γ is described geometrically but without reference to a specific parametrization. For instance, "the integral of $\overline{z}$ *counterclockwise* around the unit circle $\{z \in \mathbb{C} : |z| = 1\}$" is now a phrase that makes sense, even though we have not indicated a specific parametrization of the unit circle. Note, however, that the direction counts: The integral of $\overline{z}$ counterclockwise around the unit circle is $2\pi i$. If the direction is reversed, then the integral changes sign: The integral of $\overline{z}$ *clockwise* around the unit circle is $-2\pi i$.

2.2 Complex Differentiability and Conformality

2.2.1 Limits

Until now we have developed a complex differential and integral calculus. We now unify the notions of partial derivative and total derivative in the complex context. For convenience, we shall repeat some ideas from §1.3.

First we need a suitable notion of limit. The definition is in complete analogy with the usual definition in calculus:

Let $U \subseteq \mathbb{C}$ be open, $P \in U$, and $g : U \setminus \{P\} \to \mathbb{C}$ a function. We say that

$$\lim_{z \to P} g(z) = \ell \,, \quad \ell \in \mathbb{C} \,, \tag{2.2.1.1}$$

if for any $\epsilon > 0$ there is a $\delta > 0$ such that when $z \in U$ and $0 < |z - P| < \delta$, then $|g(z) - \ell| < \epsilon$.

2.2.2 Continuity

In a similar fashion, if f is a complex-valued function on an open set U and $P \in U$, then we say that f is *continuous* at P if $\lim_{z \to P} f(z) = f(P)$.

2.2.3 The Complex Derivative

Now let f be a function on the open set U in $\mathbb{C}$ and consider, in analogy with one variable calculus, the difference quotient

$$\frac{f(z) - f(P)}{z - P} \tag{2.2.3.1}$$

for $P \neq z \in U$. In case

$$\lim_{z \to P} \frac{f(z) - f(P)}{z - P} \tag{2.2.3.2}$$

exists (§§1.3.5), then we say that f possesses a *complex derivative* at P. We denote the complex derivative by $f'(P)$.

The classical method of studying complex function theory is by means of the complex derivative. We take this opportunity to tie up the (well-motivated) classical viewpoint with our present one.

2.2.4 Holomorphicity and the Complex Derivative

Let $U \subseteq \mathbb{C}$ be an open set and let f be holomorphic on U. Then f' exists at each point of U and

$$f'(z) = \frac{\partial f}{\partial z} \tag{2.2.4.1}$$

for all $z \in U$ (where $\partial f / \partial z$ is defined as in §§1.3.3).

Note that, as a consequence, we can (and often will) write f' for $\partial f / \partial z$ when f is holomorphic. The following result is a converse:

If $f \in C^1(U)$ and f has a complex derivative f' at each point of U, then f is holomorphic on U. In particular, if a continuous, complex-valued function f on U has a complex derivative at each point and if f' is continuous on U, then f is holomorphic on U. Such a function satisfies the Cauchy-Riemann equations (1.3.2.2).

It is perfectly logical to consider an f that possesses a complex derivative at each point of U *without* the additional assumption that $f \in C^1(U)$. It turns out that under these circumstances u and v still satisfy the Cauchy-Riemann equations. It is a deeper result, due to Goursat, that if f has a complex derivative at each point of U, then $f \in C^1(U)$ and hence f is holomorphic. See [GK], especially the Appendix on Goursat's theorem, for details.

2.2.5 Conformality

Now we make some remarks about "conformality." Stated loosely, a function is *conformal* at a point $P \in \mathbb{C}$ if the function "preserves angles" at P and "stretches equally in all directions" at P. Holomorphic functions enjoy both properties:

Let f be holomorphic in a neighborhood of $P \in \mathbb{C}$. Let w_1, w_2 be complex numbers of unit modulus. Consider the directional derivatives

$$D_{w_1} f(P) \equiv \lim_{t \to 0} \frac{f(P + tw_1) - f(P)}{t} \tag{2.2.5.1}$$

and

$$D_{w_2} f(P) \equiv \lim_{t \to 0} \frac{f(P + tw_2) - f(P)}{t}. \tag{2.2.5.2}$$

Then

(2.2.5.3) $|D_{w_1} f(P)| = |D_{w_2} f(P)|$.

(2.2.5.4) If $|f'(P)| \neq 0$, then the directed angle from w_1 to w_2 equals the directed angle from $D_{w_1} f(P)$ to $D_{w_2} f(P)$.

In fact the last statement has an important converse: If **(2.2.5.4)** holds at P, then f has a complex derivative at P. If **(2.2.5.3)** holds at P, then either f or $\overline{f}$ has a complex derivative at P. Thus a function that is conformal (in either sense) at all points of an open set U must possess the complex derivative at each point of U. By the discussion in §§2.2.4, f is therefore holomorphic if it is C^1. Or, by Goursat's theorem, it would then follow that the function is holomorphic on U, with the C^1 condition being automatic.

2.3 The Cauchy Integral Theorem and Formula

2.3.1 The Cauchy Integral Formula

Suppose that U is an open set in $\mathbb{C}$ and that f is a holomorphic function on U. Let $z_0 \in U$ and let $r > 0$ be such that $\overline{D}(P,r) \subseteq U$. Let $\gamma : [0,1] \to \mathbb{C}$ be the C^1 curve $\gamma(t) = P + r\cos(2\pi t) + ir\sin(2\pi t)$. Then, for each $z \in D(P,r)$,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{2.3.1.1}$$

2.3.2 The Cauchy Integral Theorem, Basic Form

If f is a holomorphic function on an open disc U in the complex plane, and if $\gamma : [a,b] \to U$ is a C^1 curve in U with $\gamma(a) = \gamma(b)$, then

$$\oint_\gamma f(z)\, dz = 0. \tag{2.3.2.1}$$

An important converse of Cauchy's theorem is called *Morera's theorem*:

> Let f be a continuous function on a connected open set $U \subseteq \mathbb{C}$. If
>
> $$\oint_\gamma f(z)\, dz = 0 \tag{2.3.2.2}$$
>
> for every simple, closed curve γ in U, then f is holomorphic on U.

In the statement of Morera's theorem, the phrase "every simple, closed curve" may be replaced by "every triangle" or "every square" or "every circle."

2.3.3 More General Forms of the Cauchy Theorems

Now we present the very useful general statements of the Cauchy integral theorem and formula. First we need a piece of terminology. A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be *piecewise* C^k if

$$[a,b] = [a_0,a_1] \cup [a_1,a_2] \cup \cdots \cup [a_{m-1},a_m] \tag{2.3.3.1}$$

with $a = a_0 < a_1 < \cdots a_m = b$ and $\gamma\big|_{[a_{j-1},a_j]}$ is C^k for $1 \leq j \leq m$. In other words, γ is piecewise C^k if it consists finitely many C^k curves chained end to end.

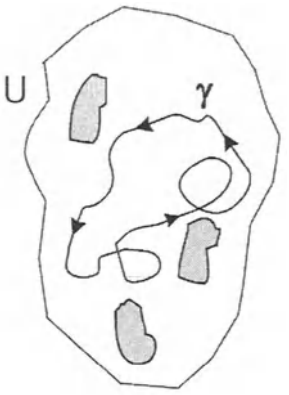

FIGURE 2.3
A curve on which the Cauchy integral theorem is valid.

Cauchy Integral Theorem: Let $f : U \to \mathbb{C}$ be holomorphic with $U \subseteq \mathbb{C}$ an open set. Then

$$\oint_\gamma f(z)\,dz = 0 \tag{2.3.3.2}$$

for each piecewise C^1 closed curve γ in U that can be deformed in U through closed curves to a point in U—see Figure 2.3.

Cauchy Integral Formula: Suppose that $\overline{D}(z,r) \subseteq U$. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z) \tag{2.3.3.3}$$

for any piecewise C^1 closed curve γ in $U \setminus \{z\}$ that can be continuously deformed in $U\setminus\{z\}$ to $\partial D(z,r)$ equipped with counterclockwise orientation. Refer to Figure 2.4.

A topological notion that is special to complex analysis is simple connectivity. We say that a domain $U \subseteq \mathbb{C}$ is *simply connected* if any closed curve in U can be continuously deformed to a point. Simple connectivity is a mathematically rigorous condition that corresponds to the intuitive notion that the region U has no holes. If U is simply connected, and γ is a closed curve in U, then it follows that γ can be continuously deformed to lie inside a disc in U. It follows that Cauchy's theorem applies to γ. To summarize: on a simply connected region, Cauchy's theorem applies (without any further hypotheses) to any closed curve in γ. Likewise, in a simply connected U, Cauchy's integral formula applied to any simple, closed curve that is oriented counterclockwise and to any point z that is inside that curve.

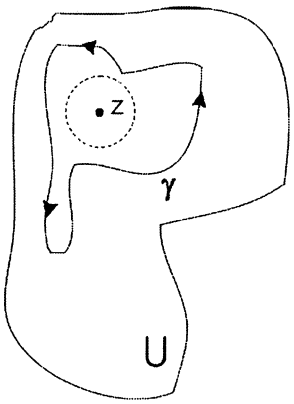

FIGURE 2.4
The Cauchy integral formula.

2.3.4 Deformability of Curves

A central fact about the complex line integral is the deformability of curves. Let $\gamma : [a, b] \to U$ be a piecewise C^1 curve in a region U of the complex plane. Let f be a holomorphic function on U. The value of the complex line integral

$$\oint_\gamma f(z)\, dz \tag{2.3.4.1}$$

does not change if the curve γ is smoothly deformed within the region U. Note that, in order for this statement to be valid, the curve γ must remain inside the region of holomorphicity U of f while it is being deformed, and it must remain a closed curve while it is being deformed. Figure 2.5 shows curves γ_1, γ_2 that *can* be deformed to one another, and a curve γ_3 that can be deformed to neither of the first two (because of the hole inside γ_3).

2.4 A Coda on the Limitations of The Cauchy Integral Formula

If f is any continuous function on the boundary of the unit disc $D = D(0, 1)$, then the Cauchy integral

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

defines a holomorphic function $F(z)$ on D (use Morera's theorem, for example, to confirm this assertion). What does the new function F have to

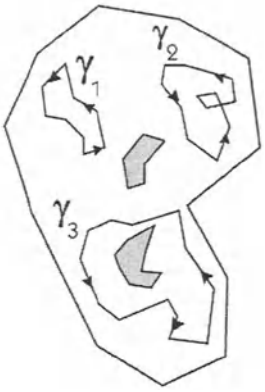

FIGURE 2.5
Deformability of curves.

do with the original function f? In general, not much.

For example, if $f(\zeta) = \overline{\zeta}$, then $F(z) \equiv 0$ (exercise). In no sense is the original function f any kind of "boundary limit" of the new function F. The question of which functions f are "natural boundary functions" for holomorphic functions F (in the sense that F is a continuous extension of F to the closed disc) is rather subtle. Its answer is well understood, but is best formulated in terms of Fourier series and the so-called Hilbert transform. The complete story is given in [KRA4].

Contrast this situation for holomorphic function with the much more succinct and clean situation for harmonic functions (§7.3).

Chapter 3

Applications of the Cauchy Theory

3.1 The Derivatives of a Holomorphic Function

3.1.1 A Formula for the Derivative

Let $U \subseteq \mathbb{C}$ be an open set and let f be holomorphic on U. Then $f \in C^\infty(U)$. Moreover, if $\overline{D}(P,r) \subseteq U$ and $z \in D(P,r)$, then

$$\left(\frac{\partial}{\partial z}\right)^k f(z) = \frac{k!}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta, \quad k = 0, 1, 2, \ldots. \tag{3.1.1.1}$$

3.1.2 The Cauchy Estimates

If f is a holomorphic on a region containing the closed disc $\overline{D}(P,r)$ and if $|f| \leq M$ on $\overline{D}(P,r)$, then

$$\left|\frac{\partial^k}{\partial z^k} f(P)\right| \leq \frac{M \cdot k!}{r^k}. \tag{3.1.2.1}$$

3.1.3 Entire Functions and Liouville's Theorem

A function f is said to be *entire* if it is defined and holomorphic on all of $\mathbb{C}$, i.e., $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. For instance, any holomorphic polynomial is entire, e^z is entire, and $\sin z, \cos z$ are entire. The function $f(z) = 1/z$ is not entire because it is undefined at $z = 0$. [In a sense that we shall make precise later (§4.1, ff.), this last function has a "singularity" at 0.] The question we wish to consider is: "Which entire functions are bounded?" This question has a very elegant and complete answer as follows:

Liouville's Theorem A bounded entire function is constant.

The reason that Liouville's theorem is true is so compelling that we include it.

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Proof: Let f be entire and assume that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix a $P \in \mathbb{C}$ and let $r > 0$. We apply the Cauchy estimate (3.1.2.1) for $k = 1$ on $\overline{D}(P,r)$. So

$$\left|\frac{\partial}{\partial z} f(P)\right| \leq \frac{M \cdot 1!}{r}. \tag{3.1.3.1}$$

Since this inequality is true for every $r > 0$, we conclude that

$$\frac{\partial f}{\partial z}(P) = 0.$$

Since P was arbitrary, we conclude that

$$\frac{\partial f}{\partial z} \equiv 0.$$

Therefore f is constant. □

The reasoning that establishes Liouville's theorem can also be used to prove this more general fact: If $f : \mathbb{C} \to \mathbb{C}$ is an entire function and if for some real number C and some positive integer k, it holds that

$$|f(z)| \leq C \cdot (1 + |z|)^k$$

for all z, then f is a polynomial in z of degree at most k.

3.1.4 The Fundamental Theorem of Algebra

One of the most elegant applications of Liouville's Theorem is a proof of what is known as the Fundamental Theorem of Algebra (see also §§1.1.7):

> **The Fundamental Theorem of Algebra:** Let $p(z)$ be a non-constant (holomorphic) polynomial. Then p has a root. That is, there exists an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Proof: Suppose not. Then $g(z) = 1/p(z)$ is entire. Also when $|z| \to \infty$, then $|p(z)| \to +\infty$. Thus $1/|p(z)| \to 0$ as $|z| \to \infty$; hence g is bounded. By Liouville's Theorem, g is constant, hence p is constant. Contradiction. □

If, in the theorem, p has degree $k \geq 1$, then let α_1 denote the root provided by the Fundamental Theorem. By the Euclidean algorithm (see [HUN]), we may divide $z - \alpha_1$ into p with no remainder to obtain

$$p(z) = (z - \alpha_1) \cdot p_1(z). \tag{3.1.4.1}$$

Here p_1 is a polynomial of degree $k-1$. If $k-1 \geq 1$, then, by the theorem, p_1 has a root α_2 . Thus p_1 is divisible by $(z-\alpha_2)$ and we have

$$p(z) = (z-\alpha_1)\cdot(z-\alpha_2)\cdot p_2(z) \tag{3.1.4.2}$$

for some polynomial $p_2(z)$ of degree $k-2$. This process can be continued until we arrive at a polynomial p_k of degree 0; that is, p_k is constant. We have derived the following fact: If $p(z)$ is a holomorphic polynomial of degree k, then there are k complex numbers $\alpha_1, \ldots \alpha_k$ (not necessarily distinct) and a non-zero constant C such that

$$p(z) = C\cdot(z-\alpha_1)\cdots(z-\alpha_k). \tag{3.1.4.3}$$

If some of the roots of p coincide, then we say that p has *multiple roots.* To be specific, if m of the values $\alpha_{j_1}, \ldots, \alpha_{j_m}$ are equal to some complex number α, then we say that p has a root of order m at α (or that p has a root of *multiplicity* m at α). An example will make the idea clear: Let

$$p(z) = (z-5)^3\cdot(z+2)^8\cdot(z-1)\cdot(z+6).$$

Then we say that p has a root of order 3 at 5, a root of order 8 at -2, and it has roots of order 1 at 1 and at -6. We also say that p has *simple roots* at 1 and -6.

3.1.5 Sequences of Holomorphic Functions and their Derivatives

A sequence of functions g_j defined on a common domain E is said to *converge uniformly* to a limit function g if, for each $\epsilon > 0$, there is a number $N > 0$ such that for all $j > N$ it holds that $|g_j(x) - g(x)| < \epsilon$ for every $x \in E$. The key point is that the degree of closeness of $g_j(x)$ to $g(x)$ is independent of $x \in E$.

Let $f_j : U \to \mathbb{C}$, $j = 1, 2, 3 \ldots$, be a sequence of holomorphic functions on an open set U in $\mathbb{C}$. Suppose that there is a function $f : U \to \mathbb{C}$ such that, for each compact subset E (a compact set is one that is closed and bounded—see Figure 3.1) of U, the restricted sequence $f_j|_E$ converges uniformly to $f|_E$. Then f is holomorphic on U. [In particular, $f \in C^\infty(U)$.]

If f_j, f, U are as in the preceding paragraph, then, for any $k \in \{0, 1, 2, \ldots\}$, we have

$$\left(\frac{\partial}{\partial z}\right)^k f_j(z) \to \left(\frac{\partial}{\partial z}\right)^k f(z) \tag{3.1.5.1}$$

uniformly on compact sets.

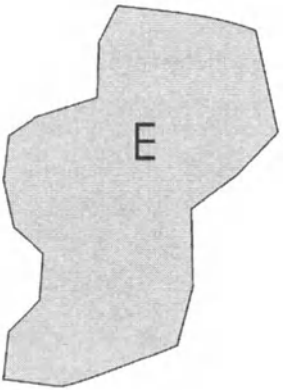

FIGURE 3.1
A compact set is closed and bounded.

3.1.6 The Power Series Representation of a Holomorphic Function

The ideas being considered in this section can be used to develop our understanding of power series. A *power series*

$$\sum_{j=0}^{\infty} a_j(z-P)^j \tag{3.1.6.1}$$

is defined to be the limit of its *partial sums*

$$S_N(z) = \sum_{j=0}^{N} a_j(z-P)^j. \tag{3.1.6.2}$$

We say that the partial sums *converge* to the sum of the entire series.

Any given power series has a *disc of convergence.* More precisely, let

$$r = \frac{1}{\limsup_{j\to\infty} |a_j|^{1/j}}. \tag{3.1.6.3}$$

The power series (3.1.6.2) will then certainly converge on the disc $D(P,r)$; the convergence will be absolute and uniform on any disc $\overline{D}(P,r')$ with $r' < r$.

For clarity, we should point out that in many examples the sequence $|a_j|^{1/j}$ actually converges as $j \to \infty$. Then we may take r to be equal to $1/\lim_{j\to\infty} |a_j|^{1/j}$. The reader should be aware, however, that in case the sequence $\{|a_j|^{1/j}\}$ does not converge, then one must use the more formal definition (3.1.6.3) of r. See [KRA2], [RUD1].

Of course the partial sums, being polynomials, are holomorphic on *any* disc $D(P,r)$. If the disc of convergence of the power series is $D(P,r)$, then let f denote the function to which the power series converges. Then for any $0 < r' < r$ we have that

$$S_N(z) \to f(z),$$

uniformly on $\overline{D}(P,r')$. We can conclude immediately that $f(z)$ is holomorphic on $D(P,r)$. Moreover, we know that

$$\left(\frac{\partial}{\partial z}\right)^k S_N(z) \to \left(\frac{\partial}{\partial z}\right)^k f(z). \tag{3.1.6.4}$$

This shows that a differentiated power series has a disc of convergence at least as large as the disc of convergence (with the same center) of the original series, and that the differentiated power series converges on that disc to the derivative of the sum of the original series. In fact, the differentiated series has exactly the same radius of convergence as the original.

The most important fact about power series for complex function theory is this: If f is a holomorphic function on a domain $U \subseteq \mathbb{C}$, if $P \in U$, and if the disc $D(P,r)$ lies in U, then f may be represented as a convergent power series on $D(P,r)$. Explicitly, we have

$$f(z) = \sum_{j=0}^{\infty} a_j (z-P)^j, \tag{3.1.6.5}$$

where

$$a_j = \frac{f^{(j)}(P)}{j!}. \tag{3.1.6.6}$$

[Here the exponent $^{(j)}$ on f denotes the j^{th} derivative.] Thus we have an explicit way of calculating the power series expansion of any holomorphic function f about a point P of its domain, and we have an a priori knowledge of the disc on which the power series representation will converge.

3.1.7 Table of Elementary Power Series

The table on the following page presents a summary of elementary power series expansions.

3.2 The Zeros of a Holomorphic Function

3.2.1 The Zero Set of a Holomorphic Function

Let f be a holomorphic function. If f is not identically zero, then it turns out that f cannot vanish at too many points. This once again bears out the dictum that holomorphic functions are a lot like polynomials. To give this notion a precise formulation, we need to recall the topological notion of connectedness (§1.1.5). An open set $W \subseteq \mathbb{C}$ is *connected* if it is not possible to find two disjoint, non-empty open sets U, V in $\mathbb{C}$ such that

$$W = (U \cap W) \cup (V \cap W). \tag{3.2.1.1}$$

Table of Elementary Power Series

Function	Power Series abt. 0	Disc of Convergence
$\frac{1}{1-z}$	$\sum_{n=0}^{\infty} z^n$	$\{z : \|z\| < 1\}$
$\frac{1}{(1-z)^2}$	$\sum_{n=1}^{\infty} n z^{n-1}$	$\{z : \|z\| < 1\}$
$\cos z$	$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$	all z
$\sin z$	$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$	all z
e^z	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$	all z
$\log(z+1)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1}$	$\{z : \|z\| < 1\}$
$(z+1)^\beta$	$\sum_{n=0}^{\infty} \binom{\beta}{n} z^n$	$\{z : \|z\| < 1\}$

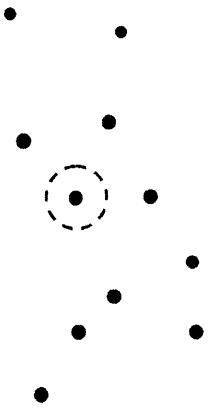

FIGURE 3.2
A discrete set.

[In the special context of open sets in the plane, it turns out that connectedness is equivalent to the condition that any two points of W may be connected by a curve that lies entirely in W—see the discussion in §§1.1.5 on path-connectedness.] Now we have:

Discreteness of the Zeros of a Holomorphic Function

> Let $W \subseteq \mathbb{C}$ be a connected (§§1.1.5) open set and let $f : W \to \mathbb{C}$ be holomorphic. Let the zero set of f be $\mathcal{Z} = \{z \in W : f(z) = 0\}$. If there are a $z_0 \in \mathcal{Z}$ and $\{z_j\}_{j=1}^{\infty} \subseteq \mathcal{Z} \setminus \{z_0\}$ such that $z_j \to z_0$, then $f \equiv 0$.

Let us formulate the result in topological terms. We recall that a point z_0 is said to be an *accumulation point* of a set $\mathcal{Z}$ if there is a sequence $\{z_j\} \subseteq \mathcal{Z} \setminus \{z_0\}$ with $\lim_{j\to\infty} z_j = z_0$. Then the theorem is equivalent to the statement: If $f : W \to \mathbb{C}$ is a holomorphic function on a connected (§§1.1.5) open set W and if $\mathcal{Z} = \{z \in W : f(z) = 0\}$ has an accumulation point *in* W, then $f \equiv 0$.

3.2.2 Discrete Sets and Zero Sets

There is still more terminology concerning the zero set of a holomorphic function in §§3.2.1. A set S is said to be *discrete* if for each $s \in S$ there is an $\epsilon > 0$ such that $D(s, \epsilon) \cap S = \{s\}$. See Figure 3.2. People also say, in a slight abuse of language, that a discrete set has points that are "isolated" or that S contains only "isolated points." The result in §§3.2.1 thus asserts that if f is a non-constant holomorphic function on a connected open set, then its zero set is discrete or, less formally, the zeros of f are isolated.

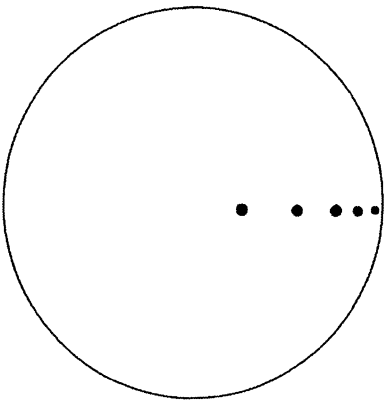

FIGURE 3.3
A zero set with a boundary accumulation point.

It is important to realize that the result in §§3.2.1 does *not* rule out the possibility that the zero set of f can have accumulation points in $\mathbb{C} \setminus U$; in particular, a non-constant holomorphic function on an open set U can indeed have zeros accumulating at a point of ∂U. Consider, for instance, the function $f(z) = \sin(1/[1-z])$ on the unit disc. The zeros of this f include $\{1 - 1/[j\pi]\}$, and these accumulate at the boundary point 1. See Figure 3.3.

3.2.3 Uniqueness of Analytic Continuation

A consequence of the preceding basic fact (3.2.1) about the zeros of a holomorphic function is this: Let $U \subseteq \mathbb{C}$ be a connected open set and $D(P,r) \subseteq U$. If f is holomorphic on U and $f\big|_{D(P,r)} \equiv 0$, then $f \equiv 0$ on U.

Here are some further corollaries:

(3.2.3.1) Let $U \subseteq \mathbb{C}$ be a connected open set. Let f, g be holomorphic on U. If $\{z \in U : f(z) = g(z)\}$ has an accumulation point in U, then $f \equiv g$.

(3.2.3.2) Let $U \subseteq \mathbb{C}$ be a connected open set and let f, g be holomorphic on U. If $f \cdot g \equiv 0$ on U, then either $f \equiv 0$ on U or $g \equiv 0$ on U.

(3.2.3.3) Let $U \subseteq \mathbb{C}$ be connected and open and let f be holomorphic on U. If there is a $P \in U$ such that

$$\left(\frac{\partial}{\partial z}\right)^j f(P) = 0 \tag{3.2.3.3.1}$$

for every $j \in \{0, 1, 2, \ldots\}$, then $f \equiv 0$.

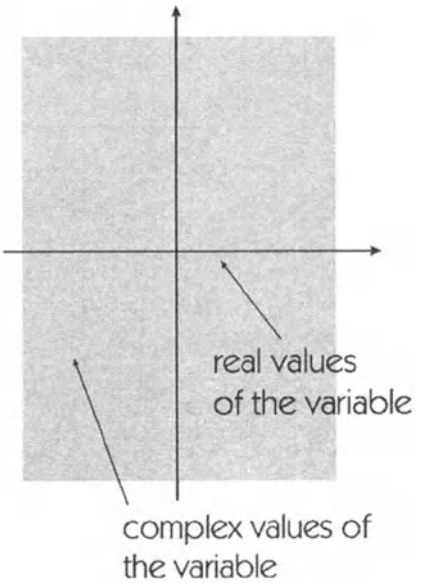

FIGURE 3.4
The principle of persistence of functional relations.

(3.2.3.4) If f and g are entire holomorphic functions and if $f(x) = g(x)$ for all $x \in \mathbb{R} \subseteq \mathbb{C}$, then $f \equiv g$. It also holds that functional identities that are true for all real values of the variable are also true for complex values of the variable (Figure 3.4). For instance,

$$\sin^2 z + \cos^2 z = 1 \qquad \text{for all } z \in \mathbb{C} \tag{3.2.3.4.1}$$

because the identity is true for all $z = x \in \mathbb{R}$. This is an instance of the "principle of persistence of functional relations"—see [GK].

Chapter 4

Isolated Singularities and Laurent Series

4.1 The Behavior of a Holomorphic Function near an Isolated Singularity

4.1.1 Isolated Singularities

It is often important to consider a function that is holomorphic on a punctured open set $U \setminus \{P\} \subset \mathbb{C}$. Refer to Figure 4.1.

In this chapter we shall obtain a new kind of infinite series expansion which generalizes the idea of the power series expansion of a holomorphic function about a (nonsingular) point—see §§3.1.6. We shall in the process completely classify the behavior of holomorphic functions near an isolated singular point (§§4.1.3).

4.1.2 A Holomorphic Function on a Punctured Domain

Let $U \subseteq \mathbb{C}$ be an open set and $P \in U$. Suppose that $f : U \setminus \{P\} \to \mathbb{C}$ is holomorphic. In this situation we say that f has an *isolated singular point* (or *isolated singularity*) at P. The implication of the phrase is usually just that f is defined and holomorphic on some such "deleted neighborhood" of P. The specification of the set U is of secondary interest; we wish to consider the behavior of f "near P."

4.1.3 Classification of Singularities

There are three possibilities for the behavior of f near P that are worth distinguishing:

S. G. Krantz, *Handbook of Complex Variables*
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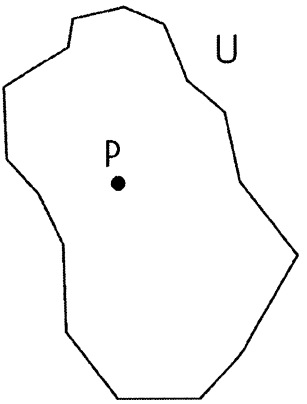

FIGURE 4.1
A punctured domain.

(4.1.3.1) $|f(z)|$ is bounded on $D(P,r)\backslash\{P\}$ for some $r > 0$ with $D(P,r) \subseteq U$; i.e., there is some $r > 0$ and some $M > 0$ such that $|f(z)| \leq M$ for all $z \in U \cap D(P,r) \setminus \{P\}$.

(4.1.3.2) $\lim_{z\to P} |f(z)| = +\infty$.

(4.1.3.3) Neither **(i)** nor **(ii)**.

4.1.4 Removable Singularities, Poles, and Essential Singularities

We shall see momentarily that, if case **(4.1.3.1)** holds, then f has a limit at P that extends f so that it is holomorphic on all of U. It is commonly said in this circumstance that f has a *removable singularity* at P. In case **(4.1.3.2)**, we will say that f has a *pole* at P. In case **(4.1.3.3)**, f will be said to have an *essential singularity* at P. Our goal in this and the next subsection is to understand **(4.1.3.1)**–**(4.1.3.2)** in some further detail.

4.1.5 The Riemann Removable Singularities Theorem

Let $f : D(P,r) \setminus \{P\} \to \mathbb{C}$ be holomorphic and bounded. Then

(4.1.5.1) $\lim_{z\to P} f(z)$ exists.

(4.1.5.2) The function $\widehat{f}: D(P,r) \to \mathbb{C}$ defined by

$$\widehat{f}(z) = \begin{cases} f(z) & \text{if } z \neq P \\ \lim\limits_{\zeta \to P} f(\zeta) & \text{if } z = P \end{cases}$$

is holomorphic.

4.1.6 The Casorati-Weierstrass Theorem

If $f: D(P,r_0) \setminus \{P\} \to \mathbb{C}$ is holomorphic and P is an essential singularity of f, then $f(D(P,r) \setminus \{P\})$ is dense in $\mathbb{C}$ for any $0 < r < r_0$.

Now we have seen that, at a removable singularity P, a holomorphic function f on $D(P,r_0) \setminus \{P\}$ can be continued to be holomorphic on all of $D(P,r_0)$. And, near an essential singularity at P, a holomorphic function g on $D(P,r_0) \setminus \{P\}$ has image that is dense in $\mathbb{C}$. The third possibility, that h has a *pole* at P, has yet to be described. This case will be examined further in the next section.

We next develop a new type of doubly infinite series that will serve as a tool for understanding isolated singularities—especially poles.

4.2 Expansion around Singular Points

4.2.1 Laurent Series

A *Laurent series* on $D(P,r)$ is a (formal) expression of the form

$$\sum_{j=-\infty}^{+\infty} a_j (z-P)^j. \tag{4.2.1.1}$$

Note that the individual terms are each defined for all $z \in D(P,r) \setminus \{P\}$.

4.2.2 Convergence of a Doubly Infinite Series

To discuss convergence of Laurent series, we must first make a general agreement as to the meaning of the convergence of a "doubly infinite" series $\sum_{j=-\infty}^{+\infty} \alpha_j$. We say that such a series *converges* if $\sum_{j=0}^{+\infty} \alpha_j$ and $\sum_{j=1}^{+\infty} \alpha_{-j} = \sum_{j=-\infty}^{-1} \alpha_j$ converge in the usual sense. In this case, we set

$$\sum_{-\infty}^{+\infty} \alpha_j = \left(\sum_{j=0}^{+\infty} \alpha_j \right) + \left(\sum_{j=1}^{+\infty} \alpha_{-j} \right). \tag{4.2.2.1}$$

We can now present the analogues for Laurent series of our basic results for power series.

4.2.3 Annulus of Convergence

The set of convergence of a Laurent series is either an open set of the form $\{z : 0 \leq r_1 < |z - P| < r_2\}$, together with perhaps some or all of the boundary points of the set, *or* a set of the form $\{z : 0 \leq r_1 < |z - P| < +\infty\}$, together with perhaps some or all of the boundary points of the set. Such an open set is called an *annulus* centered at P. We shall let

$$D(P, +\infty) = \{z : |z - P| < +\infty\} = \mathbb{C}, \tag{4.2.3.1}$$

$$D(P, 0) = \{z : |z - P| < 0\} = \emptyset, \tag{4.2.3.2}$$

and

$$\overline{D}(P, 0) = \{P\}. \tag{4.2.3.3}$$

As a result, all (open) annuli (plural of "annulus") can be written in the form

$$D(P, r_2) \setminus \overline{D}(P, r_1), \quad 0 \leq r_1 \leq r_2 \leq +\infty. \tag{4.2.3.4}$$

In precise terms, the "domain of convergence" of a Laurent series is given as follows:

Let

$$\sum_{j=-\infty}^{+\infty} a_j (z - P)^j$$

be a doubly infinite series. There are unique nonnegative extended real numbers (4.2.3.4) r_1 and r_2 (r_1 or r_2 may be $+\infty$) such that the series converges absolutely for all z with $r_1 < |z - P| < r_2$ and diverges for z with $|z - P| < r_1$ or $|z - P| > r_2$. Also, if $r_1 < s_1 \leq s_2 < r_2$, then $\sum_{j=-\infty}^{+\infty} |a_j (z - P)^j|$ converges uniformly on $\{z : s_1 \leq |z - P| \leq s_2\}$ and, consequently, $\sum_{j=-\infty}^{+\infty} a_j (z - P)^j$ converges absolutely and uniformly there.

4.2.4 Uniqueness of the Laurent Expansion

Let $0 \leq r_1 < r_2 \leq \infty$. If the Laurent series $\sum_{j=-\infty}^{+\infty} a_j (z - P)^j$ converges on $D(P, r_2) \setminus \overline{D}(P, r_1)$ to a function f, then, for any r satisfying $r_1 < r < r_2$, and each $j \in \mathbb{Z}$,

$$a_j = \frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - P)^{j+1}} \, d\zeta. \tag{4.2.4.1}$$

In particular, the a_j's are uniquely determined by f.

We turn now to establishing that convergent Laurent expansions of functions holomorphic on an annulus do in fact exist.

4.2.5 The Cauchy Integral Formula for an Annulus

Suppose that $0 \leq r_1 < r_2 \leq +\infty$ and that $f : D(P, r_2) \setminus \overline{D}(P, r_1) \to \mathbb{C}$ is holomorphic. Then, for each s_1, s_2 such that $r_1 < s_1 < s_2 < r_2$ and each $z \in D(P, s_2) \setminus \overline{D}(P, s_1)$, it holds that

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - P| = s_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \oint_{|\zeta - P| = s_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{4.2.5.1}$$

4.2.6 Existence of Laurent Expansions

Now we have our main result:

If $0 \leq r_1 < r_2 \leq \infty$ and $f : D(P, r_2) \setminus \overline{D}(P, r_1) \to \mathbb{C}$ is holomorphic, then there exist complex numbers a_j such that

$$\sum_{j=-\infty}^{+\infty} a_j (z - P)^j \tag{4.2.6.1}$$

converges on $D(P, r_2) \setminus \overline{D}(P, r_1)$ to f. If $r_1 < s_1 < s_2 < r_2$, then the series converges absolutely and uniformly on $\overline{D}(P, s_2) \setminus D(P, s_1)$.

The series expansion is independent of s_1 and s_2. In fact, for each fixed $j = 0, \pm 1, \pm 2, \ldots$, the value of

$$a_j = \frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - P)^{j+1}}\, d\zeta \tag{4.2.6.2}$$

is independent of r provided that $r_1 < r < r_2$.

4.2.7 Holomorphic Functions with Isolated Singularities

Now let us specialize what we have learned about Laurent series expansions to the case of $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ holomorphic, that is, to a holomorphic function with an isolated singularity:

If $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ is holomorphic, then f has a unique Laurent series expansion

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - P)^j \tag{4.2.7.1}$$

that converges absolutely for $z \in D(P, r) \setminus \{P\}$. The convergence is uniform on compact subsets of $D(P, r) \setminus \{P\}$. The coefficients are given by

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(P, s)} \frac{f(\zeta)}{(\zeta - P)^{j+1}}\, d\zeta, \quad \text{any } 0 < s < r. \tag{4.2.7.2}$$

4.2.8 Classification of Singularities in Terms of Laurent Series

There are three mutually exclusive possibilities for the Laurent series

$$\sum_{j=-\infty}^{\infty} a_j(z-P)^j$$

about an isolated singularity P:

(4.2.8.1) $a_j = 0$ for all $j < 0$.

(4.2.8.2) For some $k \geq 1, a_j = 0$ for all $-\infty < j < -k$, but $a_k \neq 0$.

(4.2.8.3) Neither **(i)** nor **(ii)** applies.

These three cases correspond exactly to the three types of isolated singularities that we discussed in §4.1.3: case **(4.2.8.1)** occurs if and only if P is a removable singularity; case **(4.2.8.2)** occurs if and only if P is a pole (of order k); and case **(4.2.8.3)** occurs if and only if P is an essential singularity.

To put this matter in other words: In case **(4.2.8.1)**, we have a power series that converges, of course, to a holomorphic function. In case **(4.2.8.2)**, our Laurent series has the form

$$\sum_{j=-k}^{\infty} a_j(z-P)^j = (z-P)^{-k} \sum_{j=-k}^{\infty} a_j(z-P)^{j+k} = (z-P)^{-k} \sum_{j=0}^{\infty} a_{j-k}(z-P)^j.$$

Since $a_{-k} \neq 0$, we see that, for z near P, the function defined by the series behaves like $a_{-k} \cdot (z-P)^{-k}$. In short, the function (in absolute value) blows up like $|z-P|^{-k}$ as $z \to P$. A graph in $(|z|, |f(z)|)$-space would exhibit a "pole"-like singularity. This is the source of the terminology "pole." See Figure 4.2. Case **(4.2.8.3)**, corresponding to an essential singularity, is much more complicated; in this case there are infinitely many negative terms in the Laurent expansion and, by Casorati-Weierstrass (§§4.1.6), they interact in a complicated fashion.

Picard's Great Theorem (§§10.5.3) will tell us more about the behavior of a holomorphic function near an essential singularity.

4.3 Examples of Laurent Expansions

4.3.1 Principal Part of a Function

When f has a pole at P, it is customary to call the negative power part of the Laurent expansion of f around P the *principal part* of f at

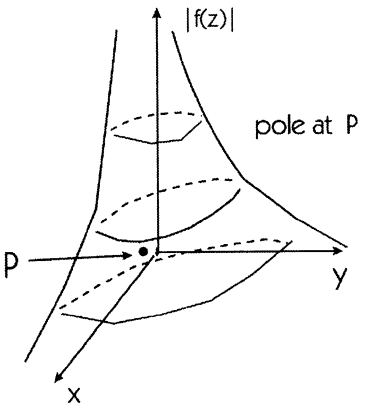

FIGURE 4.2
A pole at P.

P. (Occasionally we shall also use the terminology "Laurent polynomial.") That is, if

$$f(z) = \sum_{j=-k}^{\infty} a_j (z - P)^j \tag{4.3.1.1}$$

for z near P, then the *principal part* of f at P is

$$\sum_{j=-k}^{-1} a_j (z - P)^j. \tag{4.3.1.2}$$

As an example, the Laurent expansion about 0 of the function $f(z) = (z^2 + 1)/\sin(z^3)$ is

$$\begin{aligned} f(z) &= (z^2 + 1) \cdot \frac{1}{\sin(z^3)} \\ &= (z^2 + 1) \cdot \frac{1}{z^3} \cdot \frac{1}{1 - z^6/3! + \cdots} \\ &= \frac{1}{z^3} + \frac{1}{z} + \text{ (a holomorphic function).} \end{aligned}$$

The principal part of f is $1/z^3 + 1/z$.

For a second example, consider the function $f(z) = (z^2+2z+2)\sin(1/(z+1))$. Its Laurent expansion about the point -1 is

$$f(z) = ((z+1)^2 + 1) \cdot \left[\frac{1}{z+1} - \frac{1}{6(z+1)^3} + \frac{1}{120(z+1)^5} - \frac{1}{5040(z+1)^7} + - \cdots \right]$$

$$= (z+1) + \frac{5}{6}\frac{1}{(z+1)} - \frac{19}{120}\frac{1}{(z+1)^3} + \frac{41}{5040}\frac{1}{(z+1)^5} - + \cdots .$$

The principal part of f at the point -1 is

$$\frac{5}{6}\frac{1}{(z+1)} - \frac{19}{120}\frac{1}{(z+1)^3} + \frac{41}{5040}\frac{1}{(z+1)^5} - + \cdots .$$

4.3.2 Algorithm for Calculating the Coefficients of the Laurent Expansion

Let f be holomorphic on $D(P,r) \setminus \{P\}$ and suppose that f has a pole of order k at P. Then the Laurent series coefficients a_j of f expanded about the point P, for $j = -k, -k+1, -k+2, \ldots$, are given by the formula

$$a_j = \frac{1}{(k+j)!}\left(\frac{\partial}{\partial z}\right)^{k+j}\left((z-P)^k \cdot f\right)\Big|_{z=P}. \tag{4.3.2.1}$$

4.4 The Calculus of Residues

4.4.1 Functions with Multiple Singularities

It turns out to be useful, especially in evaluating various types of integrals, to consider functions that have more than one "singularity." We want to consider the following general question:

> Suppose that $f : U \setminus \{P_1, P_2, \ldots, P_n\} \to \mathbb{C}$ is a holomorphic function on an open set $U \subseteq \mathbb{C}$ with finitely many distinct points $P_1, P_2, \ldots, P_n$ removed. Suppose further that
>
> $$\gamma : [0,1] \to U \setminus \{P_1, P_2, \ldots, P_n\} \tag{4.4.1.1}$$
>
> is a piecewise C^1 closed curve (§2.3.3) that (typically) "surrounds" some of the points $P_1, \ldots, P_n$. Then how is $\oint_\gamma f$ related to the behavior of f near the points $P_1, P_2, \ldots, P_n$?

The first step is to restrict our attention to open sets U for which $\oint_\gamma f$ is necessarily 0 if $P_1, P_2, \ldots, P_n$ are removable singularities of f. See the next subsection.

4.4.2 The Residue Theorem

Suppose that $U \subseteq \mathbb{C}$ is a simply connected open set in $\mathbb{C}$, and that $P_1, \ldots, P_n$ are distinct points of U. Suppose that $f : U \setminus \{P_1, \ldots, P_n\} \to \mathbb{C}$

is a holomorphic function and γ is a piecewise C^1 curve in $U \setminus \{P_1, \ldots, P_n\}$. Set

$$R_j = \text{the coefficient of } (z - P_j)^{-1} \text{ in the Laurent expansion of } f \text{ about } P_j. \tag{4.4.2.1}$$

Then

$$\oint_\gamma f = \sum_{j=1}^{n} R_j \cdot \left(\oint_\gamma \frac{1}{\zeta - P_j} d\zeta \right). \tag{4.4.2.2}$$

4.4.3 Residues

The result just stated is used so often that some special terminology is commonly used to simplify its statement. First, the number R_j is usually called the *residue* of f at P_j , written $\text{Res}_f(P_j)$. Note that this terminology of considering the number R_j attached to the point P_j makes sense because $\text{Res}_f(P_j)$ is completely determined by knowing f in a small neighborhood of P_j. In particular, the value of the residue does not depend on what the other points P_k, $k \neq j$, might be, or on how f behaves near those points.

4.4.4 The Index or Winding Number of a Curve about a Point

The second piece of terminology associated to our result deals with the integrals that appear on the right-hand side of equation (4.4.2.2).

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1 closed curve and if $P \notin \widetilde{\gamma} \equiv \gamma([a, b])$, then the *index of* γ *with respect to* P, written $\text{Ind}_\gamma(P)$, is defined to be the number

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P} d\zeta. \tag{4.4.4.1}$$

The index is also sometimes called the "winding number of the curve γ about the point P." It is a fact that $\text{Ind}_\gamma(P)$ is always an integer. Figure 4.3 illustrates the index of various curves γ with respect to different points P. Intuitively, the index measures the number of times the curve wraps around P, with counterclockwise being the positive direction of wrapping and clockwise being the negative.

The fact that the index is an integer-valued function suggests that the index counts the topological winding of the curve γ. Note in particular that a curve that traces a circle about the origin k times in a counterclockwise direction has index k with respect to the origin; a curve that traces a circle about the origin k times in a clockwise direction has index $-k$ with respect to the origin.

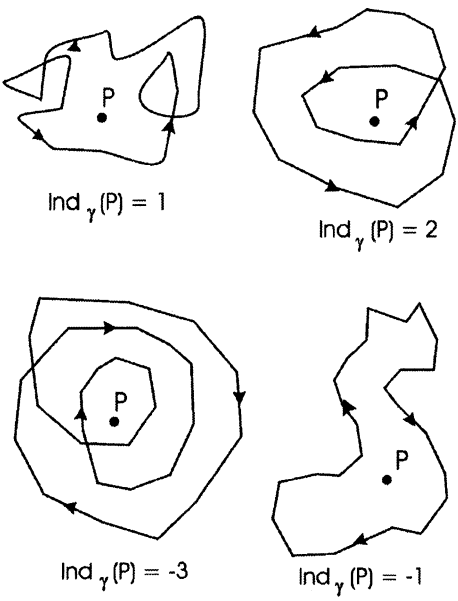

FIGURE 4.3
Examples of the index of a curve.

4.4.5 Restatement of the Residue Theorem

Using the notation of residue and index, the Residue Theorem's formula becomes

$$\oint_\gamma f = 2\pi i \cdot \sum_{j=1}^{n} \mathrm{Res}_f(P_j) \cdot \mathrm{Ind}_\gamma(P_j). \tag{4.4.5.1}$$

People sometimes state this formula informally as "the integral of f around γ equals $2\pi i$ times the sum of the residues counted according to the index of γ about the singularities."

4.4.6 Method for Calculating Residues

We need a method for calculating residues.

Let f be a function with a pole of order k at P. Then

$$\mathrm{Res}_f(P) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}\Big((z-P)^k f(z)\Big)\Big|_{z=P}. \tag{4.4.6.1}$$

This is just a special case of the formula (4.3.2.1).

Poles and Laurent Coefficients

Item	Formula
j^{th} Laurent coefficient of f with pole of order k at P	$\frac{1}{(k+j)!}\frac{d^{k+j}}{dz^{k+j}}\left[(z-P)^k \cdot f\right]\Big\|_{z=P}$
residue of f with a pole of order k at P	$\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-P)^k \cdot f\right]\Big\|_{z=P}$
order of pole of f at P	least integer $k \geq 0$ such that $(z-P)^k \cdot f$ is bounded near P
order of pole of f at P	$\lim_{z\to P}\left\|\frac{\log\|f(z)\|}{\log\|z-P\|}\right\|$

4.4.7 Summary Charts of Laurent Series and Residues

We provide two charts, the first of which summarizes key ideas about Laurent coefficients and the second of which contains key ideas about residues.

4.5 Applications to the Calculation of Definite Integrals and Sums

4.5.1 The Evaluation of Definite Integrals

One of the most classical and fascinating applications of the calculus of residues is the calculation of definite (usually improper) real integrals. It is an over-simplification to call these calculations, taken together, a "technique": it is more like a *collection* of techniques. We present several instances of the method.

Techniques for Finding the Residue at P

Function	Type of Pole	Calculation
$f(z)$	simple	$\lim_{z\to P}(z-P)\cdot f(z)$
$f(z)$	pole of order k k is the least integer such that $\lim_{z\to P}\mu(z)$ exists, where $\mu(z)=(z-P)^k f(z)$	$\lim_{z\to P}\dfrac{\mu^{(k-1)}(z)}{(k-1)!}$
$\dfrac{m(z)}{n(z)}$	$m(P)\neq 0$, $n(z)=0$, $n'(P)\neq 0$	$\dfrac{m(P)}{n'(P)}$
$\dfrac{m(z)}{n(z)}$	m has zero of order k at P n has zero of order $(k+1)$ at P	$(k+1)\cdot\dfrac{m^{(k)}(P)}{n^{(k+1)}(P)}$
$\dfrac{m(z)}{n(z)}$	m has zero of order r at P n has zero of order $(k+r)$ at P	$\lim_{z\to P}\dfrac{\mu^{(k-1)}(z)}{(k-1)!}$, $\mu(z)=(z-P)^k\dfrac{m(z)}{n(z)}$

4.5.2 A Basic Example

To evaluate

$$\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx, \tag{4.5.2.1}$$

we "complexify" the integrand to $f(z)=1/(1+z^4)$ and consider the integral

$$\oint_{\gamma_R}\frac{1}{1+z^4}dx.$$

See Figure 4.4.

Now part of the game here is to choose the right piecewise C^1 curve or "contour" γ_R. The appropriateness of our choice is justified (after the fact) by the calculation that we are about to do. Assume that $R>1$. Define

$$\gamma_R^1(t)=t+i0 \quad \text{if} \quad -R\le t\le R,$$
$$\gamma_R^2(t)=Re^{it} \quad \text{if} \quad 0\le t\le\pi.$$

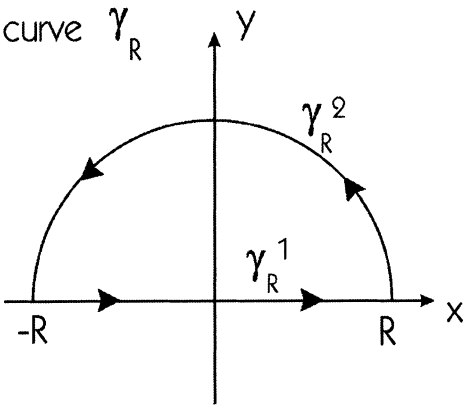

FIGURE 4.4
The curve γ_R in Subsection 4.5.2.

Call these two curves, taken together, γ or γ_R.

Now we set $U = \mathbb{C}$, $P_1 = 1/\sqrt{2} + i/\sqrt{2}$, $P_2 = -1/\sqrt{2} + i/\sqrt{2}$, $P_3 = -1/\sqrt{2} - i/\sqrt{2}$, $P_4 = 1/\sqrt{2} - i/\sqrt{2}$; the points P_1, P_2, P_3, P_4 are the poles of $1/[1 + z^4]$. Thus $f(z) = 1/(1 + z^4)$ is holomorphic on $U \setminus \{P_1, \ldots, P_4\}$ and the Residue Theorem applies.

On the one hand,

$$\oint_\gamma \frac{1}{1+z^4}\, dz = 2\pi i \sum_{j=1,2} \mathrm{Ind}_\gamma(P_j) \cdot \mathrm{Res}_f(P_j),$$

where we sum only over the poles of f that lie inside γ. These are P_1 and P_2. An easy calculation shows that

$$\mathrm{Res}_f(P_1) = \frac{1}{4(1/\sqrt{2} + i/\sqrt{2})^3} = -\frac{1}{4}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

and

$$\mathrm{Res}_f(P_2) = \frac{1}{4(-1/\sqrt{2} + i/\sqrt{2})^3} = -\frac{1}{4}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right).$$

Of course the index at each point is 1. So

$$\begin{aligned}\oint_\gamma \frac{1}{1+z^4}\, dz &= 2\pi i \left(-\frac{1}{4}\right)\left[\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right] \\ &= \frac{\pi}{\sqrt{2}}. \end{aligned} \tag{4.5.2.2}$$

On the other hand,

$$\oint_\gamma \frac{1}{1+z^4}\, dz = \oint_{\gamma_R^1} \frac{1}{1+z^4}\, dz + \oint_{\gamma_R^2} \frac{1}{1+z^4}\, dz.$$

Trivially,

$$\oint_{\gamma_R^1} \frac{1}{1+z^4}\,dz = \int_{-R}^{R} \frac{1}{1+t^4} \cdot 1 \cdot dt \to \int_{-\infty}^{\infty} \frac{1}{1+t^4} dt \qquad (4.5.2.3)$$

as $R \to +\infty$. That is good, because this last is the integral that we wish to evaluate. Better still,

$$\left| \oint_{\gamma_R^2} \frac{1}{1+z^4} dx \right| \leq \{\text{length}(\gamma_R^2)\} \cdot \max_{\gamma_R^2} \left| \frac{1}{1+z^4} \right| \leq \pi R \cdot \frac{1}{R^4 - 1}.$$

[Here we use the inequality $|1+z^4| \geq |z|^4 - 1$, as well as (2.1.7.2).] Thus

$$\left| \oint_{\gamma_R^2} \frac{1}{1+z^4}\,dz \right| \to 0 \qquad \text{as} \qquad R \to \infty. \qquad (4.5.2.4)$$

Finally, (4.5.2.2)–(4.5.2.4) taken together yield

$$\begin{aligned} \frac{\pi}{\sqrt{2}} &= \lim_{R\to\infty} \oint_{\gamma} \frac{1}{1+z^4}\,dz \\ &= \lim_{R\to\infty} \oint_{\gamma_R^1} \frac{1}{1+z^4}\,dz + \lim_{R\to\infty} \oint_{\gamma_R^2} \frac{1}{1+z^4}\,dz \\ &= \int_{-\infty}^{\infty} \frac{1}{1+t^4} dt + 0. \end{aligned}$$

This solves the problem: the value of the integral is $\pi/\sqrt{2}$.

In other problems, it will not be so easy to pick the contour so that the superfluous parts (in the above example, this would be the integral over γ_R^2) tend to zero, nor is it always so easy to prove that they *do* tend to zero. Sometimes, it is not even obvious how to complexify the integrand.

4.5.3 Complexification of the Integrand

We evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx \qquad (4.5.3.1)$$

by using the contour γ_R as in Figure 4.5 (which is the same as Figure 4.4 from the previous example). The obvious choice for the complexification of the integrand is

$$f(z) = \frac{\cos z}{1+z^2} = \frac{[e^{iz} + e^{-iz}]/2}{1+z^2} = \frac{[e^{ix}e^{-y} + e^{-ix}e^{y}]/2}{1+z^2}. \qquad (4.5.3.2)$$

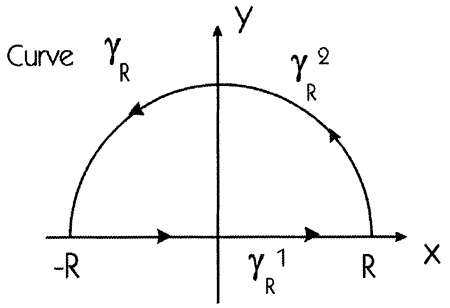

FIGURE 4.5
The curve γ_R in Subsection 4.5.3.

Now $|e^{ix}e^{-y}| = |e^{-y}| \leq 1$ on γ_R but $|e^{-ix}e^{y}| = |e^{y}|$ becomes quite large on γ_R when R is large and positive. There is no evident way to alter the contour so that good estimates result. Instead, we alter the function! Let $g(z) = e^{iz}/(1+z^2)$.

On the one hand (for $R > 1$),

$$\begin{aligned}\oint_{\gamma_R} g(z) &= 2\pi i \cdot \mathrm{Res}_g(i) \cdot \mathrm{Ind}_{\gamma_R}(i) \\ &= 2\pi i \left(\frac{1}{2ei}\right) \cdot 1 = \frac{\pi}{e}.\end{aligned}$$

On the other hand, with $\gamma_R^1(t) = t, -R \leq t \leq R$, and $\gamma_R^2(t) = Re^{it}, 0 \leq t \leq \pi$, we have

$$\oint_{\gamma_R} g(z)\,dz = \oint_{\gamma_R^1} g(z)\,dz + \oint_{\gamma_R^2} g(z)\,dz.$$

Of course

$$\oint_{\gamma_R^1} g(z)\,dz \to \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx \quad \text{as} \quad R \to \infty.$$

And

$$\left|\oint_{\gamma_R^2} g(z)\,dz\right| \leq \mathrm{length}(\gamma_R^2) \cdot \max_{\gamma_R^2} |g| \leq \pi R \cdot \frac{1}{R^2-1} \to 0 \quad \text{as} \quad R \to \infty.$$

Here we have again used (2.1.7.2).

Thus

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx = \mathrm{Re}\left(\frac{\pi}{e}\right) = \frac{\pi}{e}.$$

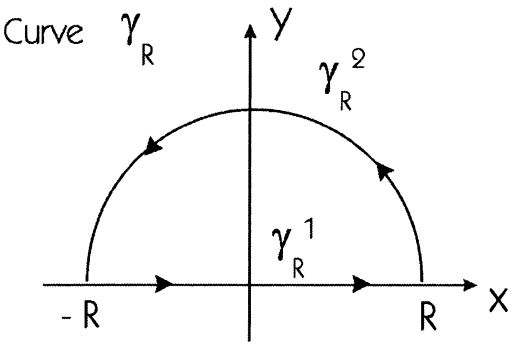

FIGURE 4.6
The curve γ_R in Subsection 4.5.4.

4.5.4 An Example with a More Subtle Choice of Contour

Let us evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx. \tag{4.5.4.1}$$

Before we begin, we remark that $\sin x/x$ is bounded near zero; also, the integral converges at ∞ (as an improper Riemann integral) by integration by parts. So the problem makes sense. Using the lesson learned from the last example, we consider the function $g(z) = e^{iz}/z$. However, the pole of e^{iz}/z is at $z = 0$ and that lies *on the contour* in Figure 4.6. Thus *that* contour may not be used. We instead use the contour $\mu = \mu_R$ that is depicted in Figure 4.7.

Define

$$\begin{aligned}
\mu_R^1(t) &= t, \qquad -R \le t \le -1/R, \\
\mu_R^2(t) &= e^{it}/R, \ \pi \le t \le 2\pi, \\
\mu_R^3(t) &= t, \qquad 1/R \le t \le R, \\
\mu_R^4(t) &= Re^{it}, \ 0 \le t \le \pi.
\end{aligned}$$

Clearly

$$\oint_\mu g(z)\, dz = \sum_{j=1}^{4} \oint_{\mu_R^j} g(z)\, dz.$$

On the one hand, for $R > 0$,

$$\oint_\mu g(z)\, dz = 2\pi i \mathrm{Res}_g(0) \cdot \mathrm{Ind}_\mu(0) = 2\pi i \cdot 1 \cdot 1 = 2\pi i. \tag{4.5.4.2}$$

On the other hand,

$$\oint_{\mu_R^1} g(z)\, dz + \oint_{\mu_R^3} g(z)\, dz \to \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx \quad \text{as} \quad R \to \infty. \tag{4.5.4.3}$$

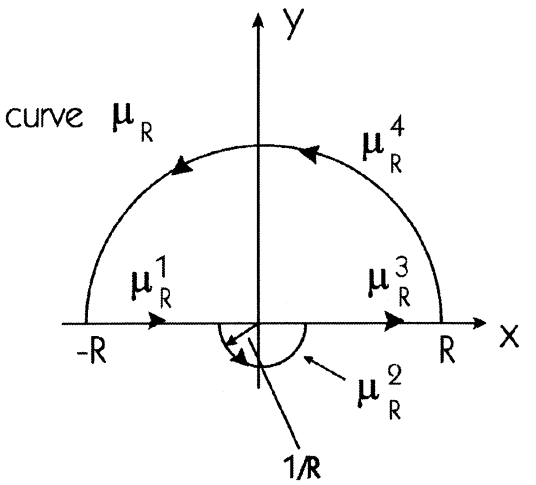

FIGURE 4.7
The curve μ_R in Subsection 4.5.4.

Furthermore,

$$\left|\oint_{\mu_R^4} g(z)\,dz\right| \le \left|\oint_{\substack{\mu_R^4 \\ \text{Im } y < \sqrt{R}}} g(z)\,dz\right| + \left|\oint_{\substack{\mu_R^4 \\ \text{Im } y \ge \sqrt{R}}} g(z)\,dz\right|$$
$$\equiv A + B.$$

Now

$$A \le \text{length}(\mu_R^4 \cap \{z : \text{Im } z < \sqrt{R}\}) \cdot \max\{|g(z)| : z \in \mu_R^4, y < \sqrt{R}\}$$
$$\le 4\sqrt{R} \cdot \left(\frac{1}{R}\right) \to 0 \quad \text{as} \quad R \to \infty.$$

Also

$$B \le \text{length}(\mu_R^4 \cap \{z : \text{Im } z \ge \sqrt{R}\}) \cdot \max\{|g(z)| : z \in \mu_R^4, y \ge \sqrt{R}\}$$
$$\le \pi R \cdot \left(\frac{e^{-\sqrt{R}}}{R}\right) \to 0 \quad \text{as} \quad R \to \infty.$$

So

$$\left|\oint_{\mu_R^4} g(z)\,dz\right| \to 0 \quad \text{as} \quad R \to \infty. \tag{4.5.4.4}$$

Finally,

$$\oint_{\mu_R^2} g(z)\,dz = \int_{\pi}^{2\pi} \frac{e^{i(e^{it}/R)}}{e^{it}/R} \cdot \left(\frac{i}{R} e^{it}\right) dt$$

$$= i \int_{\pi}^{2\pi} e^{i(e^{it}/R)} dt.$$

As $R \to \infty$ this tends to

$$= i \int_{\pi}^{2\pi} 1 \, dt$$
$$= \pi i \quad \text{as} \quad R \to \infty. \tag{4.5.4.5}$$

In summary, (4.5.4.2) – (4.5.4.5) yield

$$2\pi i = \oint_{\mu} g(z) \, dz = \sum_{j=1}^{4} \oint_{\mu_R^j} g(z) \, dz$$

$$\to \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx + \pi i \quad \text{as} \quad R \to \infty.$$

Taking imaginary parts yields

$$\pi = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

4.5.5 Making the Spurious Part of the Integral Disappear

Consider the integral

$$\int_0^{\infty} \frac{x^{1/3}}{1+x^2} dx. \tag{4.5.5.1}$$

We complexify the integrand by setting $f(z) = z^{1/3}/(1+z^2)$. Note that, on the simply connected set $U = \mathbb{C} \setminus \{iy : y < 0\}$, the expression $z^{1/3}$ is unambiguously defined as a holomorphic function by setting $z^{1/3} = r^{1/3} e^{i\theta/3}$ when $z = re^{i\theta}$, $-\pi/2 < \theta < 3\pi/2$. We use the contour displayed in Figure 4.8.

We must do this since $z^{1/3}$ is not a well-defined holomorphic function in any neighborhood of 0. Let us use the notation from the figure. We refer to the preceding examples for some of the parametrizations that we now use.

Clearly

$$\oint_{\mu_R^3} f(z) \, dz \to \int_0^{\infty} \frac{t^{1/3}}{1+t^2} dt.$$

Of course that is good, but what will become of the integral over μ_R^1? We have

$$\oint_{\mu_R^1} = \int_{-R}^{-1/R} \frac{t^{1/3}}{1+t^2} dt$$

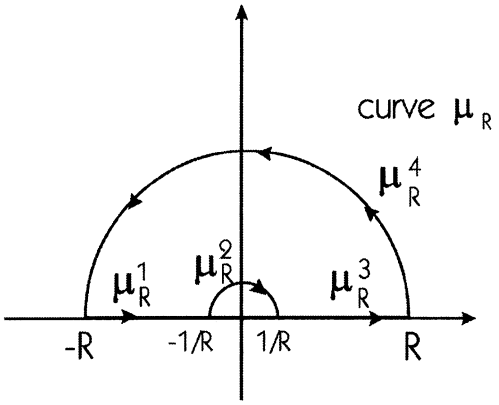

FIGURE 4.8
The curve μ_R in Subsection 4.5.5.

$$= \int_{1/R}^{R} \frac{(-t)^{1/3}}{1+t^2} dt$$
$$= \int_{1/R}^{R} \frac{e^{i\pi/3} t^{1/3}}{1+t^2} dt.$$

(by our definition of $z^{1/3}$!). Thus

$$\oint_{\mu_R^3} f(z)\,dz + \oint_{\mu_R^1} f(z)\,dz \to \left(1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\right) \int_0^\infty \frac{t^{1/3}}{1+t^2} dt \quad \text{as} \quad R \to \infty.$$

On the other hand,

$$\left| \oint_{\mu_R^4} f(z)\,dz \right| \leq \pi R \cdot \frac{R^{1/3}}{R^2 - 1} \to 0 \quad \text{as} \quad R \to \infty$$

and

$$\oint_{\mu_R^2} f(z)\,dz = \int_{-\pi}^{-2\pi} \frac{(e^{it}/R)^{1/3}}{1 + e^{2it}/R^2} (i) e^{it}/R\, dt$$
$$= R^{-4/3} \int_{-\pi}^{-2\pi} \frac{e^{i4t/3}}{1 + e^{2it}/R^2}\, dt \to 0 \quad \text{as} \quad R \to \infty.$$

So

$$\oint_{\mu_R} f(z)\,dz \to \left(\frac{3}{2} + \frac{\sqrt{3}}{2} i\right) \int_0^\infty \frac{t^{1/3}}{1+t^2}\, dt \quad \text{as} \quad R \to \infty. \tag{4.5.5.2}$$

The calculus of residues tells us that, for $R > 1$,

$$\begin{aligned}\oint_{\mu_R} f(z)\,dz &= 2\pi i \mathrm{Res}_f(i) \cdot \mathrm{Ind}_{\mu_R}(i) \\ &= 2\pi i \left(\frac{e^{i\pi/6}}{2i}\right) \cdot 1 \\ &= \pi\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right). \end{aligned} \tag{4.5.5.3}$$

Finally, (4.5.5.2) and (4.5.5.3) taken together yield

$$\int_0^\infty \frac{t^{1/3}}{1+t^2}\,dt = \frac{\pi}{\sqrt{3}}. \qquad \square$$

4.5.6 The Use of the Logarithm

While the integral

$$\int_0^\infty \frac{dx}{x^2+6x+8} \tag{4.5.6.1}$$

can be calculated using methods of calculus, it is enlightening to perform the integration by complex variable methods. Note that if we endeavor to use the integrand $f(z) = 1/(z^2+6z+8)$ together with the idea of the last example, then there is no "auxiliary radius" that helps. More precisely, $((re^{i\theta})^2+6re^{i\theta}+8)$ is a constant multiple of r^2+6r+8 only if θ is an integer multiple of 2π. The following non-obvious device is often of great utility in problems of this kind. Define $\log z$ on $U \equiv \mathbb{C} \setminus \{x : x \geq 0\}$ by $\log(re^{i\theta}) = (\log r) + i\theta$ when $0 < \theta < 2\pi, r > 0$. Here $\log r$ is understood to be the standard real logarithm. Then, on U, log is a well-defined holomorphic function. [Observe here that there are infinitely many ways to define the logarithm function on U. One could set $\log(re^{i\theta}) = (\log r) + i(\theta + 2k\pi)$ for any integer choice of k. What we have done here is called "choosing a branch" of the logarithm.]

We use the contour η_R displayed in Figure 4.9 and integrate the function $g(z) = \log z/(z^2+6z+8)$. Let

$$\begin{aligned}\eta_R^1(t) &= t + i/\sqrt{2R}, \quad 1/\sqrt{2R} \leq t \leq R, \\ \eta_R^2(t) &= Re^{it}, \quad \theta_0 \leq t \leq 2\pi - \theta_0,\end{aligned}$$

where $\theta_0(R) = \tan^{-1}(1/(R\sqrt{2R}))$

$$\begin{aligned}\eta_R^3(t) &= R - t - i/\sqrt{2R}, \quad 0 \leq t \leq R - 1/\sqrt{2R}, \\ \eta_R^4(t) &= e^{-it}/\sqrt{R}, \quad \pi/4 \leq t \leq 7\pi/4.\end{aligned}$$

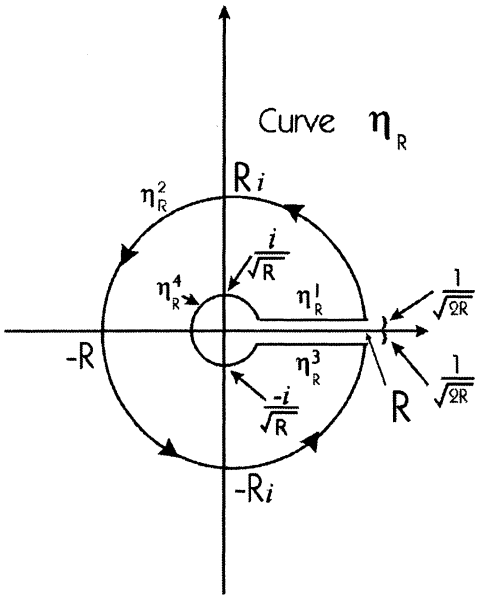

FIGURE 4.9
The curve η_R in Subsection 4.5.6.

Now

$$\begin{aligned}\oint_{\eta_R} g(z)\,dz &= 2\pi i(\mathrm{Res}_{\eta_R}(-2)\cdot 1 + \mathrm{Res}_{\eta_R}(-4)\cdot 1)\\ &= 2\pi i\left(\frac{\log(-2)}{2} + \frac{\log(-4)}{-2}\right)\\ &= 2\pi i\left(\frac{\log 2 + \pi i}{2} + \frac{\log 4 + \pi i}{-2}\right)\\ &= -\pi i \log 2. \end{aligned} \tag{4.5.6.2}$$

Also, it is straightforward to check that

$$\left|\oint_{\eta_R^2} g(z)\,dz\right| \to 0, \tag{4.5.6.3}$$

$$\left|\oint_{\eta_R^4} g(z)\,dz\right| \to 0,$$

as $R \to \infty$. The device that makes this technique work is that, as $R \to \infty$,

$$\log(x + i/\sqrt{2R}) - \log(x - i/\sqrt{2R}) \to -2\pi i.$$

So

$$\oint_{\eta_R^1} g(z)\,dz + \oint_{\eta_R^3} g(z)\,dz \to -2\pi i \int_0^\infty \frac{dt}{t^2+6t+8}. \tag{4.5.6.4}$$

Now (4.5.6.2)–(4.5.6.4) taken together yield

$$\int_0^\infty \frac{dt}{t^2+6t+8} = \frac{1}{2}\log 2.$$

4.5.7 Summing a Series Using Residues

We sum the series

$$\sum_{j=1}^{\infty} \frac{x}{j^2\pi^2 - x^2} \tag{4.5.7.1}$$

using contour integration. Define $\cot z = \cos z/\sin z$. For $n = 1, 2, \ldots$ let Γ_n be the contour (shown in Figure 4.10) consisting of the counterclockwise oriented square with corners $\{(\pm 1 \pm i)\cdot(n+\frac{1}{2})\cdot\pi\}$. For z fixed and $n > |z|$ we calculate using residues that

$$\frac{1}{2\pi i}\oint_{\Gamma_n} \frac{\cot\zeta}{\zeta(\zeta - z)}d\zeta = \sum_{j=1}^{n} \frac{1}{j\pi(j\pi - z)} + \sum_{j=1}^{n} \frac{1}{j\pi(j\pi + z)} + \frac{\cot z}{z} - \frac{1}{z^2}.$$

When $n \gg |z|$, it is easy to estimate the left-hand side in modulus by

$$\left(\frac{1}{2\pi}\right)\cdot[4(2n+1)\pi]\cdot\left(\frac{C}{n(n-|z|)}\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus we see that

$$\sum_{j=1}^{\infty} \frac{1}{j\pi(j\pi - z)} + \sum_{j=1}^{\infty} \frac{1}{j\pi(j\pi + z)} = -\frac{\cot z}{z} + \frac{1}{z^2}.$$

We conclude that

$$\sum_{j=1}^{\infty} \frac{2}{j^2\pi^2 - z^2} = -\frac{\cot z}{z} + \frac{1}{z^2}$$

or

$$\sum_{j=1}^{\infty} \frac{z}{j^2\pi^2 - z^2} = -\frac{1}{2}\cot z + \frac{1}{2z}.$$

This is the desired result . □

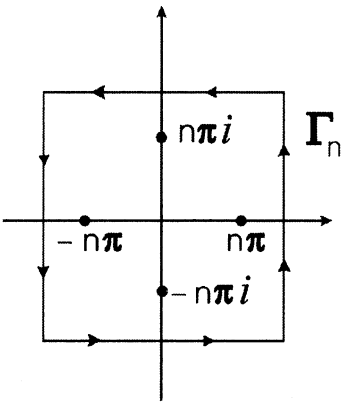

FIGURE 4.10
The curve Γ_n in Subsection 4.5.7.

4.5.8 Summary Chart of Some Integration Techniques

On the next page we present, in chart form, just a few of the key methods of using residues to evaluate definite integrals.

4.6 Meromorphic Functions and Singularities at Infinity

4.6.1 Meromorphic Functions

We have considered carefully those functions that are holomorphic on sets of the form $D(P,r) \setminus \{P\}$ or, more generally, of the form $U \setminus \{P\}$, where U is an open set in $\mathbb{C}$ and $P \in U$. As we have seen in our discussion of the calculus of residues, sometimes it is important to consider the possibility that a function could be "singular" at more than just one point. The appropriate precise definition requires a little preliminary consideration of what kinds of sets might be appropriate as "sets of singularities."

4.6.2 Discrete Sets and Isolated Points

A set S in $\mathbb{C}$ is *discrete* if and only if for each $z \in S$ there is a positive number r (depending on z) such that

$$S \cap D(z,r) = \{z\}. \tag{4.6.2.1}$$

We also say in this circumstance that S consists of isolated points. This latter terminology is something of an abuse of language since being discrete

is a property of the set S, not of the individual points in S.

4.6.3 Definition of Meromorphic Function

Now fix an open set U; we next define the central concept of meromorphic function on U.

A *meromorphic function* f on U *with singular set* S is a function $f : U \setminus S \to \mathbb{C}$ such that

(4.6.3.1) S is discrete;

(4.6.3.2) f is holomorphic on $U \setminus S$ (note that $U \setminus S$ is necessarily open in $\mathbb{C}$);

(4.6.3.3) for each $P \in S$ and $r > 0$ such that $D(P,r) \subseteq U$ and $S \cap D(P,r) = \{P\}$, the function $f|_{D(P,r)\setminus\{P\}}$ has a (finite order) pole at P.

For convenience, one often suppresses explicit consideration of the set S and just says that f is a meromorphic function on U. Sometimes we say, informally, that a meromorphic function on U is a function on U that is holomorphic "except for poles." Implicit in this description is the idea that a pole is an "isolated singularity." In other words, a point P is a pole of f if and only if there is a disc $D(P,r)$ around P such that f is holomorphic on $D(P,r) \setminus \{P\}$ and has a pole at P. Back on the level of precise language, we see that our definition of a meromorphic function on U implies that, for each $P \in U$, either there is a disc $D(P,r) \subseteq U$ such that f is holomorphic on $D(P,r)$ *or* there is a disc $D(P,r) \subseteq U$ such that f is holomorphic on $D(P,r) \setminus \{P\}$ and has a pole at P.

4.6.4 Examples of Meromorphic Functions

Meromorphic functions are very natural objects to consider, primarily because they result from considering the (algebraic) reciprocals of holomorphic functions:

If U is a connected open set in $\mathbb{C}$ and if $f : U \to \mathbb{C}$ is a holomorphic function with $f \not\equiv 0$, then the function

$$F : U \setminus \{z : f(z) = 0\} \to \mathbb{C} \tag{4.6.4.1}$$

defined by $F(z) = 1/f(z)$ is a meromorphic function on U with singular set (or pole set) equal to $\{z \in U : f(z) = 0\}$. In a sense that can be made precise, all meromorphic functions arise as *quotients* of holomorphic functions.

Use of Residues to Evaluate Integrals

Integral	Properties of	Value of Integral
$I = \int_{-\infty}^{\infty} f(x)\,dx$	No poles of $f(z)$ on real axis. Finite number of poles of $f(z)$ in plane. $\lvert f(z)\rvert \le \frac{C}{\lvert z\rvert^2}$ for z large.	$I = 2\pi i \times$ (sum of residues of f in upper half plane)
$I = \int_{-\infty}^{\infty} f(x)\,dx$	$f(z)$ may have simple poles on real axis. Finite number of poles of $f(z)$ in plane. $\lvert f(z)\rvert \le \frac{C}{\lvert z\rvert^2}$ for z large.	$I = 2\pi i \times$ (sum of residues of f in upper half plane) $+ \pi i \times$ (sum of residues of $f(z)$ on real axis
$I = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$	p, q polynomials. $[\deg p] + 2 \le \deg q$. q has no real zeros.	$I = 2\pi i \times$ (sum of residues of $p(z)/q(z)$ in upper half plane)
$I = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$	p, q polynomials. $[\deg p] + 2 \le \deg q$. $p(z)/q(z)$ may have simple poles on real axis.	$I = 2\pi i \times$ (sum of residues of $p(z)/q(z)$ in upper half plane) $+ \pi i \times$ (sum of residues of $p(z)/q(z)$ on real axis

Use of Residues to Evaluate Integrals, Continued

Integral	Properties of	Value of Integral
$I = \int_{-\infty}^{\infty} e^{i\alpha x} \cdot f(x)\,dx$	$\alpha > 0$, z large $\lvert f(z)\rvert \leq \frac{C}{\lvert z\rvert}$ No poles of f on real axis.	$I = 2\pi i \times$ (sum of residues of $e^{i\alpha z} f(z)$ in upper half plane)
$I = \int_{-\infty}^{\infty} e^{i\alpha x} \cdot f(x)\,dx$	$\alpha > 0$, z large $\lvert f(z)\rvert \leq \frac{C}{\lvert z\rvert}$ $f(z)$ may have simple poles on real axis	$I = 2\pi i \times$ (sum of residues of $e^{i\alpha z} f(z)$ in upper half plane) $+ \pi i \times$ (sum of residues of $e^{i\alpha z} f(z)$ on real axis)

4.6.5 Meromorphic Functions with Infinitely Many Poles

It is quite possible for a meromorphic function on an open set U to have infinitely many poles in U. The function $1/\sin(1/z)$ is an obvious example on $U = D \setminus \{0\}$.

4.6.6 Singularities at Infinity

Our discussion so far of singularities of holomorphic functions can be generalized to include the limit behavior of holomorphic functions as $|z| \to +\infty$. This is a powerful method with many important consequences. Suppose for example that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. We can associate to f a new function $G : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by setting $G(z) = f(1/z)$. The behavior of the function G near 0 reflects, in an obvious sense, the behavior of f as $|z| \to +\infty$. For instance

$$\lim_{|z| \to +\infty} |f(z)| = +\infty \tag{4.6.6.1}$$

if and only if G has a pole at 0.

Suppose that $f : U \to \mathbb{C}$ is a holomorphic function on an open set $U \subseteq \mathbb{C}$ and that, for some $R > 0, U \supseteq \{z : |z| > R\}$. Define $G : \{z : 0 < |z| <$

$1/R\} \to \mathbb{C}$ by $G(z) = f(1/z)$. Then we say that

(4.6.6.2) f has a *removable singularity* at ∞ if G has a removable singularity at 0.

(4.6.6.3) f has a *pole at* ∞ if G has a pole at 0.

(4.6.6.4) f has an *essential singularity* at ∞ if G has an essential singularity at 0.

4.6.7 The Laurent Expansion at Infinity

The Laurent expansion of G around 0, $G(z) = \sum_{-\infty}^{+\infty} a_n z^n$, yields immediately a series expansion for f which converges for $|z| > R$, namely,

$$f(z) \equiv G(1/z) = \sum_{-\infty}^{+\infty} a_n z^{-n} = \sum_{-\infty}^{+\infty} a_{-n} z^n. \tag{4.6.7.1}$$

The series $\sum_{-\infty}^{+\infty} a_{-n} z^n$ is called the *Laurent expansion of* f *around* ∞. It follows from our definitions and from our earlier discussions that f has a removable singularity at ∞ if and only if the Laurent series of f at ∞ has no *positive* powers of z with non-zero coefficients. Also f has a pole at ∞ if and only if the series has only a finite number of positive powers of z with non-zero coefficients. Finally, f has an essential singularity at ∞ if and only if the series has infinitely many positive powers.

4.6.8 Meromorphic at Infinity

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Then $\lim_{|z| \to +\infty} |f(z)| = +\infty$ if and only if f is a nonconstant polynomial. In other words, an entire function that is not a polynomial will have an essential singularity at infinity.

The entire function f has a removable singularity at ∞ if and only if f is a constant.

Suppose that f is a meromorphic function defined on an open set $U \subseteq \mathbb{C}$ such that, for some $R > 0$, we have $U \supseteq \{z : |z| > R\}$. We say that f is *meromorphic* at ∞ if the function $G(z) \equiv f(1/z)$ is meromorphic in the usual sense on $\{z : |z| < 1/R\}$.

4.6.9 Meromorphic Functions in the Extended Plane

The definition of "meromorphic at ∞" as given is equivalent to requiring that, for some $R' > R$, f has no poles in $\{z \in \mathbb{C} : R' < |z| < \infty\}$ *and* that f has a pole at ∞.

A meromorphic function f on $\mathbb{C}$ which is also meromorphic at ∞ must be a rational function (that is, a quotient of polynomials in z). Conversely, every rational function is meromorphic on $\mathbb{C}$ and at ∞.

Remark: It is conventional to rephrase the ideas just presented by saying that the only functions that are meromorphic in the "extended plane" are rational functions. We will say more about the extended plane in §§6.3.1–6.3.3.

Chapter 5

The Argument Principle

5.1 Counting Zeros and Poles

5.1.1 Local Geometric Behavior of a Holomorphic Function

In this chapter, we shall be concerned with questions that have a geometric, qualitative nature rather than an analytical, quantitative one. These questions center around the issue of the local geometric behavior of a holomorphic function.

5.1.2 Locating the Zeros of a Holomorphic Function

Suppose that $f : U \to \mathbb{C}$ is a holomorphic function on a connected, open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P,r) \subseteq U$. We know from the Cauchy integral formula that the values of f on $D(P,r)$ are completely determined by the values of f on $\partial D(P,r)$. In particular, the number and even the location of the zeros of f in $D(P,r)$ are determined in principle by f on $\partial D(P,r)$. But it is nonetheless a pleasant surprise that there is a *simple formula* for the number of zeros of f in $D(P,r)$ in terms of f (and f') on $\partial D(P,r)$. In order to obtain a precise formula, we shall have to agree to count zeros according to multiplicity (see §§3.1.4). We now explain the precise idea.

Let $f : U \to \mathbb{C}$ be holomorphic as before, and assume that f has *some* zeros in U but that f is not identically zero. Fix $z_0 \in U$ such that $f(z_0) = 0$. Since the zeros of f are isolated, there is an $r > 0$ such that $\overline{D}(z_0,r) \subseteq U$ and such that f does not vanish on $\overline{D}(z_0,r) \setminus \{z_0\}$.

Now the power series expansion of f about z_0 has a first non-zero term determined by the least positive integer n such that $f^{(n)}(z_0) \neq 0$. (Note that $n \geq 1$ since $f(z_0) = 0$ by hypothesis.) Thus the power series expansion

S. G. Krantz, *Handbook of Complex Variables*

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of f about z_0 *begins* with the n^{th} term:

$$f(z) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^j. \tag{5.1.2.1}$$

Under these circumstances we say that f has a zero of *order* n (or *multiplicity* n) at z_0. When $n = 1$, then we also say that z_0 is a *simple* zero of f.

5.1.3 Zero of Order n

The concept of zero of "order n," or "multiplicity n," for a function f is so important that a variety of terminology has grown up around it (see also §§3.1.4). It has already been noted that when the multiplicity $n = 1$, then the zero is sometimes called *simple*. For arbitrary n, we sometimes say that "n is the order of z_0 as a zero of f." More generally if $f(z_0) = \beta$ so that, for some $n \geq 1$, the function $f(\cdot) - \beta$ has a zero of order n at z_0, then we say either that "f assumes the value β at z_0 to order n" or that "the order of the value β at z_0 is n." When $n > 1$, then we call z_0 a *multiple point* of the function f.

The next result provides a method for computing the multiplicity n of the zero at z_0 from the values of f, f' on the boundary of a disc centered at z_0.

5.1.4 Counting the Zeros of a Holomorphic Function

If f is holomorphic on a neighborhood of a disc $\overline{D}(P,r)$ and has a zero of order n at P and no other zeros in the closed disc, then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = n. \tag{5.1.4.1}$$

More generally, we consider the case that f has several zeros—with different locations and different multiplicities—inside a disc: Suppose that $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P,r) \subseteq U$. Suppose that f is non-vanishing on $\partial D(P,r)$ and that $z_1, z_2, \ldots, z_k$ are the zeros of f in the interior of the disc. Let n_ℓ be the order of the zero of f at z_ℓ, $\ell = 1, \ldots, k$. Then

$$\frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{\ell=1}^{k} n_\ell. \tag{5.1.4.2}$$

Refer to Figure 5.1 for illustrations of both these situations.

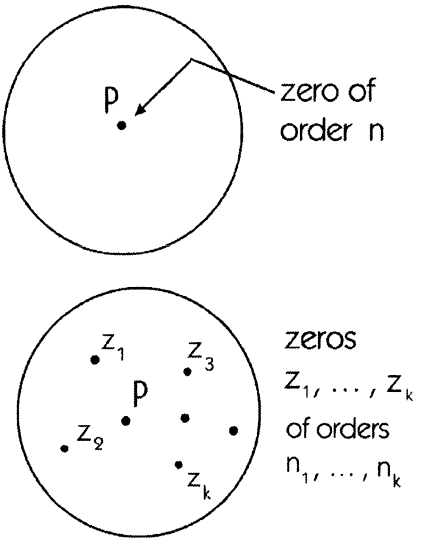

FIGURE 5.1
Locating the zeros of a holomorphic function.

5.1.5 The Argument Principle

This last formula, which is often called the *argument principle*, is both useful and important. For one thing, there is no obvious reason why the integral in the formula should be an integer, much less the crucial integer that it is. Since it is an integer, it is a counting function; and we need to learn more about it.

The integral

$$\frac{1}{2\pi i}\oint_{|\zeta - P| = r} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta \tag{5.1.5.1}$$

can be reinterpreted as follows: Consider the C^1 closed curve

$$\gamma(t) = f(P + re^{it}), \quad t \in [0, 2\pi]. \tag{5.1.5.2}$$

Then

$$\frac{1}{2\pi i}\oint_{|\zeta - P| = r} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)} dt, \tag{5.1.5.3}$$

as you can check by direct calculation. The expression on the right is just the index of the curve γ with respect to 0 (with the notion of index that we defined earlier—§§4.4.4). See Figure 5.2. Thus the number of zeros of f (counting multiplicity) inside the circle $\{\zeta : |\zeta - P| = r\}$ is equal to the index of γ with respect to the origin. This, intuitively speaking, is equal to the number of times that the f-image of the boundary circle winds around 0 in $\mathbb{C}$.

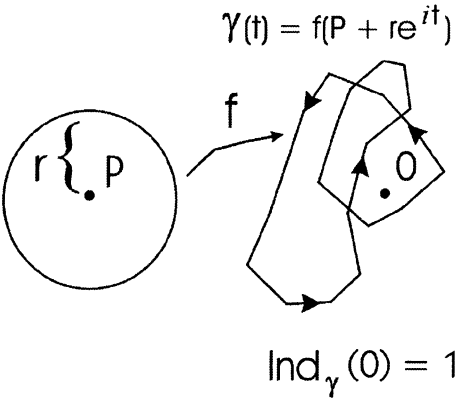

FIGURE 5.2
The argument principle: counting the zeros.

The argument principle can be extended to yield information about meromorphic functions, too. We can see that there is hope for this notion by investigating the analog of the argument principle for a pole.

5.1.6 Location of Poles

If $f : U \setminus \{Q\} \to \mathbb{C}$ is a nowhere-zero holomorphic function on $U \setminus \{Q\}$ with a pole of order n at Q and if $\overline{D}(Q,r) \subseteq U$, then

$$\frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = -n. \tag{5.1.6.1}$$

5.1.7 The Argument Principle for Meromorphic Functions

Just as with the argument principle for holomorphic functions, this new argument principle gives a counting principle for zeros and poles of meromorphic functions:

Suppose that f is a meromorphic function on an open set $U \subseteq \mathbb{C}$, that $\overline{D}(P,r) \subseteq U$, and that f has neither poles nor zeros on $\partial D(P,r)$. Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k, \tag{5.1.7.1}$$

where $n_1, n_2, \ldots, n_p$ are the multiplicities of the zeros $z_1, z_2, \ldots, z_p$ of f in $D(P,r)$ and $m_1, m_2, \ldots, m_q$ are the orders of the poles $w_1, w_2, \ldots, w_q$ of f in $D(P,r)$.

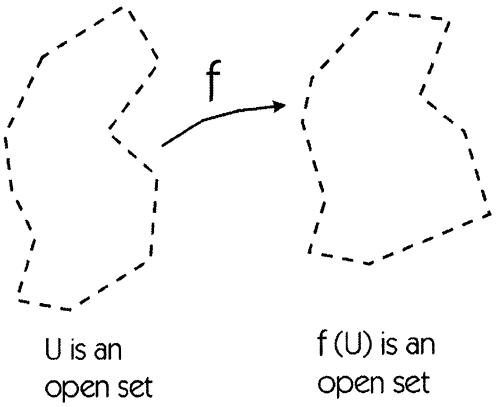

FIGURE 5.3
The open mapping principle.

5.2 The Local Geometry of Holomorphic Functions

5.2.1 The Open Mapping Theorem

The argument principle for holomorphic functions has a consequence that is one of the most important facts about holomorphic functions considered as geometric mappings:

> If $f : U \to \mathbb{C}$ is a non-constant holomorphic function on a connected open set U, then $f(U)$ is an open set in $\mathbb{C}$.
>
> (5.2.1.1)

See Figure 5.3. The result says, in particular, that if $U \subseteq \mathbb{C}$ is connected and open and if $f : U \to \mathbb{C}$ is holomorphic, then either $f(U)$ is a connected open set (the non-constant case) or $f(U)$ is a single point.

In the subject of topology, a function f is defined to be continuous if the inverse image of any open set under f is also open. In contexts where the $\epsilon - \delta$ definition makes sense, the $\epsilon - \delta$ definition (§§2.2.1, 2.2.2) is equivalent to the inverse-image-of-open-sets definition. By contrast, functions for which the direct image of any open set is open are called "open mappings."

Here is a quantitative, or counting, statement that comes from the proof of the open mapping principle: Suppose that $f : U \to \mathbb{C}$ is a non-constant holomorphic function on a connected open set U such that $P \in U$ and $f(P) = Q$ with order k. Then there are numbers $\delta, \epsilon > 0$ such that each

$q \in D(Q,\epsilon) \setminus \{Q\}$ has exactly k distinct pre-images in $D(P,\delta)$ and each pre-image is a simple point of f.

The considerations that establish the open mapping principle can also be used to establish the fact that if $f : U \to V$ is a one-to-one and onto holomorphic function, then $f^{-1} : V \to U$ is also holomorphic.

5.3 Further Results on the Zeros of Holomorphic Functions

5.3.1 Rouché's Theorem

Now we consider global aspects of the argument principle.

Suppose that $f, g : U \to \mathbb{C}$ are holomorphic functions on an open set $U \subseteq \mathbb{C}$. Suppose also that $\overline{D}(P,r) \subseteq U$ and that, for each $\zeta \in \partial D(P,r)$,

$$|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|. \tag{5.3.1.1}$$

Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta. \tag{5.3.1.2}$$

That is, the number of zeros of f in $D(P,r)$ counting multiplicities equals the number of zeros of g in $D(P,r)$ counting multiplicities.

Remark: Rouché's theorem is often stated with the stronger hypothesis that

$$|f(\zeta) - g(\zeta)| < |g(\zeta)| \tag{5.3.1.3}$$

for $\zeta \in \partial D(P,r)$. Rewriting this hypothesis as

$$\left| \frac{f(\zeta)}{g(\zeta)} - 1 \right| < 1, \tag{5.3.1.4}$$

we see that it says that the image γ under f/g of the circle $\partial D(P,r)$ lies in the disc $D(1,1)$. See Figure 5.4. Our weaker hypothesis that $|f(\zeta)-g(\zeta)| < |f(\zeta)| + |g(\zeta)|$ has the geometric interpretation that $f(\zeta)/g(\zeta)$ lies in the set $\mathbb{C} \setminus \{x + i0 : x \leq 0\}$. Either hypothesis implies that the image of the circle $\partial D(P,r)$ under f has the same "winding number" around 0 as does the image under g of that circle.

5.3.2 Typical Application of Rouché's Theorem

EXAMPLE 5.3.2.1 Let us determine the number of roots of the polynomial $f(z) = z^7 + 5z^3 - z - 2$ in the unit disc. We do so by comparing

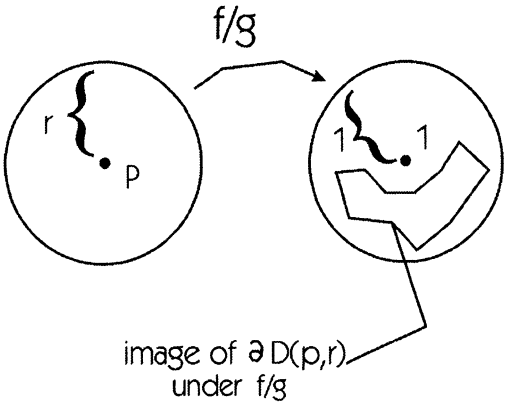

FIGURE 5.4
Rouché's theorem.

the function f to the holomorphic function $g(z) = 5z^3$ on the unit circle. For $|z| = 1$ we have

$$|f(z) - g(z)| = |z^7 - z - 2| \leq 4 < |g(\zeta)| \leq |f(\zeta)| + |g(\zeta)|.$$

By Rouché's theorem, f and g have the same number of zeros, counting multiplicity, in the unit disc. Since g has three zeros, so does f. □

5.3.3 Rouché's Theorem and the Fundamental Theorem of Algebra

Rouché's theorem provides a useful way to locate approximately the zeros of a holomorphic function that is too complicated for the zeros to be obtained explicitly. As an illustration, we analyze the zeros of a non-constant polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0. \tag{5.3.3.1}$$

If R is sufficiently large (say $R > \max\{1, n \cdot \max_{0 \leq j \leq n-1} |a_j|\}$) and $|z| = R$, then

$$\frac{|a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0|}{|z^n|} < 1. \tag{5.3.3.2}$$

Thus Rouché's theorem applies on $\overline{D}(0, R)$ with $f(z) = z^n$ and $g(z) = P(z)$. We conclude that the number of zeros of $P(z)$ inside $D(0, R)$, counting multiplicities, is the same as the number of zeros of z^n inside $D(0, R)$, counting multiplicities—namely n. Thus we recover the Fundamental Theorem of Algebra. Incidentally, this example underlines the importance of counting zeros with multiplicities: the function z^n has only one root in the naïve

sense of counting the number of points where it is zero; but it has n roots when they are counted with multiplicity.

5.3.4 Hurwitz's Theorem

A second useful consequence of the argument principle is the following result about the limit of a sequence of zero-free holomorphic functions:

> **Hurwitz's theorem** Suppose that $U \subseteq \mathbb{C}$ is a connected open set and that $\{f_j\}$ is a sequence of nowhere-vanishing holomorphic functions on U. If the sequence $\{f_j\}$ converges uniformly on compact subsets of U to a (necessarily holomorphic) limit function f_0, then either f_0 is nowhere-vanishing or $f_0 \equiv 0$.

5.4 The Maximum Principle

5.4.1 The Maximum Modulus Principle

A *domain* in $\mathbb{C}$ is a connected open set (§§1.3.1). A *bounded domain* is a connected open set U such that there is an $R > 0$ with $|z| < R$ for all $z \in U$—or $U \subseteq D(0, R)$.

The Maximum Modulus Principle

> Let $U \subseteq \mathbb{C}$ be a domain. Let f be a holomorphic function on U. If there is a point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$, then f is constant.

Here is a sharper variant of the theorem:

> Let $U \subseteq \mathbb{C}$ be a domain and let f be a holomorphic function on U. If there is a point $P \in U$ at which $|f|$ has a *local maximum*, then f is constant.

5.4.2 Boundary Maximum Modulus Theorem

The following version of the maximum principle is intuitively appealing, and is frequently useful.

> Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Then the maximum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must in fact occur on ∂U.

In other words,

$$\max_{\overline{U}} |f| = \max_{\partial U} |f|.$$

5.4.3 The Minimum Principle

Holomorphic functions (or, more precisely, their moduli) *can* have interior minima. The function $f(z) = z^2$ on $D(0,1)$ has the property that $z = 0$ is a global minimum for $|f|$. However, it is not accidental that this minimum value is 0:

> Let f be holomorphic on a domain $U \subseteq \mathbb{C}$. Assume that f never vanishes. If there is a point $P \in U$ such that $|f(P)| \leq |f(z)|$ for all $z \in U$, then f is constant. This result is proved by applying the maximum principle to the function $1/f$.

There is also a boundary minimum principle:

> Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Assume that f never vanishes on $\overline{U}$. Then the minimum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must occur on ∂U.
>
> In other words,
>
> $$\min_{\overline{U}} |f| = \min_{\partial U} |f|.$$

5.4.4 The Maximum Principle on an Unbounded Domain

It should be noted that the maximum modulus theorem is not always true on an unbounded domain. The standard example is the function $f(z) = \exp(\exp(z))$ on the domain $U = \{z = x + iy : -\pi/2 < y < \pi/2\}$. Check for yourself that $|f| = 1$ on the boundary of U. But the restriction of f to the real number line is unbounded at infinity. The theorem does, however, remain true with some additional restrictions. The result known as the Phragmen-Lindelöf theorem is one method of treating maximum modulus theorems on unbounded domains (see [RUD2]).

5.5 The Schwarz Lemma

This section treats certain estimates that bounded holomorphic functions on the unit disc necessarily satisfy. We present the classical, analytic viewpoint in the subject (instead of the geometric viewpoint—see [KRA1]).

5.5.1 Schwarz's Lemma

Let f be holomorphic on the unit disc. Assume that

(5.5.1.1) $|f(z)| \leq 1$ for all z.

(5.5.1.2) $f(0) = 0$.

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

If either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation: $f(z) \equiv \alpha z$ for some complex constant α of unit modulus.

Schwarz's lemma enables one to classify the invertible holomorphic self-maps of the unit disc (see [GK]). (Here a *self-map* of a domain U is a mapping $F : U \to U$ of the domain to itself.) These are commonly referred to as the "conformal self-maps" of the disc. The classification is as follows: If $0 \leq \theta < 2\pi$, then define the *rotation through angle* θ to be the function $\rho_\theta(z) = e^{i\theta}z$; if a is a complex number of modulus less than one, then define the associated *Möbius transformation* to be $\varphi_a(z) = [z - a]/[1 - \overline{a}z]$. Any conformal self-map of the disc is the composition of some rotation ρ_θ with some Möbius transformation φ_a. This topic is treated in detail in §6.2.

We conclude this section by presenting a generalization of the Schwarz lemma, in which we consider holomorphic mappings $f : D \to D$, but we discard the hypothesis that $f(0) = 0$. This result is known as the Schwarz-Pick lemma.

5.5.2 The Schwarz-Pick Lemma

Let f be holomorphic on the unit disc. Assume that

(5.5.2.1) $|f(z)| \leq 1$ for all z.

(5.5.2.2) $f(a) = b$ for some $a, b \in D(0, 1)$.

Then

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}. \tag{5.5.2.3}$$

Moreover, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then

$$\left|\frac{b_2 - b_1}{1 - \overline{b}_1 b_2}\right| \leq \left|\frac{a_2 - a_1}{1 - \overline{a}_1 a_2}\right|. \tag{5.5.2.4}$$

There is a "uniqueness" result in the Schwarz-Pick Lemma. If either

$$|f'(a)| = \frac{1 - |b|^2}{1 - |a|^2} \quad \text{or} \quad \left|\frac{b_2 - b_1}{1 - \overline{b}_1 b_2}\right| = \left|\frac{a_2 - a_1}{1 - \overline{a}_1 a_2}\right| \quad \text{with } a_1 \neq a_2\,, \tag{5.5.2.5}$$

then the function f is a conformal self-mapping (one-to-one, onto holomorphic function) of $D(0, 1)$ to itself.

Chapter 6

The Geometric Theory of Holomorphic Functions

6.1 The Idea of a Conformal Mapping

6.1.1 Conformal Mappings

The main objects of study in this chapter are holomorphic functions $h : U \to V$, with U and V open in $\mathbb{C}$, that are one-to-one and onto. Such a holomorphic function is called a *conformal* (or *biholomorphic*) mapping. The fact that h is supposed to be one-to-one implies that h' is nowhere zero on U [remember that if h' vanishes to order $k \geq 0$ at a point $P \in U$, then h is $(k+1)$-to-1 in a small neighborhood of P—see §§5.2.1]. As a result, $h^{-1} : V \to U$ is also holomorphic—as we discussed in §§5.2.1. A conformal map $h : U \to V$ from one open set to another can be used to transfer holomorphic functions on U to V and vice versa: that is, $f : V \to \mathbb{C}$ is holomorphic if and only if $f \circ h$ is holomorphic on U; and $g : U \to \mathbb{C}$ is holomorphic if and only if $g \circ h^{-1}$ is holomorphic on V.

Thus, if there is a conformal mapping from U to V, then U and V are essentially indistinguishable from the viewpoint of complex function theory. On a practical level, one can often study holomorphic functions on a rather complicated open set by first mapping that open set to some simpler open set, then transferring the holomorphic functions as indicated.

6.1.2 Conformal Self-Maps of the Plane

The simplest open subset of $\mathbb{C}$ is $\mathbb{C}$ itself. Thus it is natural to begin our study of conformal mappings by considering the conformal mappings of $\mathbb{C}$ to itself. In fact the conformal mappings from $\mathbb{C}$ to $\mathbb{C}$ can be explicitly described as follows:

S. G. Krantz, *Handbook of Complex Variables*

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A function $f : \mathbb{C} \to \mathbb{C}$ is a conformal mapping if and only if there are complex numbers a, b with $a \neq 0$ such that

$$f(z) = az + b \quad , \quad z \in \mathbb{C}. \tag{6.1.2.1}$$

One aspect of the result is fairly obvious: If $a, b \in \mathbb{C}$ and $a \neq 0$, then the map $z \mapsto az + b$ is certainly a conformal mapping of $\mathbb{C}$ to $\mathbb{C}$. In fact one checks easily that $z \mapsto (z - b)/a$ is the inverse mapping. The interesting part of the assertion is that these are in fact the only conformal maps of $\mathbb{C}$ to $\mathbb{C}$.

A generalization of this result about conformal maps of the plane is the following (consult §§4.6.8 as well as the detailed explanation in [GK]):

If $h : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that

$$\lim_{|z| \to +\infty} |h(z)| = +\infty,$$

then h is a polynomial.

6.2 Conformal Mappings of the Unit Disc

6.2.1 Conformal Self-Maps of the Disc

In this section we describe the set of all conformal maps of the unit disc to itself. Our first step is to determine those conformal maps of the disc to the disc that fix the origin. Let D denote the unit disc.

Let us begin by examining a conformal mapping $f : D \to D$ of the unit disc to itself and such that $f(0) = 0$. We are assuming that f is one-to-one and onto. Then, by Schwarz's lemma (§§5.5.1), $|f'(0)| \leq 1$. This reasoning applies to f^{-1} as well, so that $|(f^{-1})'(0)| \leq 1$ or $|f'(0)| \geq 1$. We conclude that $|f'(0)| = 1$. By the uniqueness part of the Schwarz lemma, f must be a rotation. So there is a complex number ω with $|\omega| = 1$ such that

$$f(z) \equiv \omega z \quad \forall z \in D. \tag{6.2.1.1}$$

It is often convenient to write a rotation as

$$\rho_\theta(z) \equiv e^{i\theta} z,$$

where we have set $\omega = e^{i\theta}$ with $0 \leq \theta < 2\pi$.

We will next generalize this result to conformal self-maps of the disc that do not necessarily fix the origin.

6.2.2 Möbius Transformations

Construction of Möbius Transformations For $a \in \mathbb{C}$, $|a| < 1$, we define

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z}. \tag{6.2.2.1}$$

Then each φ_a is a conformal self-map of the unit disc.

To see this assertion, note that if $|z| = 1$, then

$$|\varphi_a(z)| = \left|\frac{z-a}{1-\overline{a}z}\right| = \left|\frac{\overline{z}(z-a)}{1-\overline{a}z}\right| = \left|\frac{1-a\overline{z}}{1-\overline{a}z}\right| = 1. \tag{6.2.2.2}$$

Thus φ_a takes the boundary of the unit disc to itself. Since $\varphi_a(0) = -a \in D$, we conclude that φ_a maps the unit disc to itself. The same reasoning applies to $(\varphi_a)^{-1} = \varphi_{-a}$, hence φ_a is a one-to-one conformal map of the disc to the disc.

The biholomorphic self-mappings of D can now be completely characterized.

6.2.3 Self-Maps of the Disc

Let $f : D \to D$ be a holomorphic function. Then f is a conformal self-map of D if and only if there are complex numbers a, ω with $|\omega| = 1, |a| < 1$ such that

$$f(z) = \omega \cdot \varphi_a(z) \quad \forall z \in D. \tag{6.2.3.1}$$

In other words, any conformal self-map of the unit disc to itself is the composition of a Möbius transformation with a rotation.

It can also be shown that any conformal self-map f of the unit disc can be written in the form

$$f(z) = \varphi_b(\eta \cdot z),$$

for some Möbius transformation φ_b and some complex number η with $|\eta| = 1$.

6.3 Linear Fractional Transformations

6.3.1 Linear Fractional Mappings

The automorphisms (that is, conformal self-mappings) of the unit disc D are special cases of functions of the form

$$z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}. \tag{6.3.1.1}$$

It is worthwhile to consider functions of this form in generality. One restriction on this generality needs to be imposed, however; if $ad - bc = 0$, then the numerator is a constant multiple of the denominator provided that the denominator is not identically zero. So if $ad - bc = 0$, then the function is either constant or has zero denominator and is nowhere defined. Thus only the case $ad - bc \neq 0$ is worth considering in detail.

A function of the form

$$z \mapsto \frac{az + b}{cz + d} \quad , \quad ad - bc \neq 0, \tag{6.3.1.2}$$

is called a *linear fractional transformation.*

Note that $(az + b)/(cz + d)$ is not necessarily defined for all $z \in \mathbb{C}$. Specifically, if $c \neq 0$, then it is undefined at $z = -d/c$. In case $c \neq 0$,

$$\lim_{z \to -d/c} \left| \frac{az + b}{cz + d} \right| = +\infty. \tag{6.3.1.3}$$

This observation suggests that one might well, for linguistic convenience, adjoin formally a "point at ∞" to $\mathbb{C}$ and consider the value of $(az+b)/(cz+d)$ to be ∞ when $z = -d/c\,(c \neq 0)$. Thus we will think of both the domain and the range of our linear fractional transformation to be $\mathbb{C} \cup \{\infty\}$ (we sometimes also use the notation $\widehat{\mathbb{C}}$ instead of $\mathbb{C} \cup \{\infty\}$). Specifically, we are led to the following alternative method for describing a linear fractional transformation.

A function $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a *linear fractional transformation* if there exists $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$, such that either

(6.3.1.4) $c = 0, d \neq 0, f(\infty) = \infty$, and $f(z) = (a/d)z + (b/d)$ for all $z \in \mathbb{C}$;

or

(6.3.1.5) $c \neq 0, f(\infty) = a/c, f(-d/c) = \infty$, and $f(z) = (az + b)/(cz + d)$ for all $z \in \mathbb{C}, z \neq -d/c$.

It is important to realize that, as before, the status of the point ∞ is entirely formal: we are just using it as a linguistic convenience, to keep track of the behavior of $f(z)$ both where it is not defined as a map on $\mathbb{C}$ and to keep track of its behavior when $|z| \to +\infty$. The justification for the particular devices used is the fact that

(6.3.1.6) $\lim_{|z| \to +\infty} f(z) = f(\infty)$ [$c = 0$; case **(6.3.1.4)** of the definition]

(6.3.1.7) $\lim_{z \to -d/c} |f(z)| = +\infty$ [$c \neq 0$; case **(6.3.1.5)** of the definition].

6.3.2 The Topology of the Extended Plane

The limit properties of f that we described in §§6.3.1 can be considered as continuity properties of f from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$ using the definition of continuity that comes from the topology on $\mathbb{C} \cup \{\infty\}$. It is easy to formulate that topology in terms of open sets. But it is also convenient to formulate that same topological structure in terms of convergence of sequences:

A sequence $\{p_i\}$ in $\mathbb{C} \cup \{\infty\}$ *converges to* $p_0 \in \mathbb{C} \cup \{\infty\}$ (notation $\lim_{i \to \infty} p_i = p_0$) if either

(6.3.2.1) $p_0 = \infty$ and $\lim_{i \to +\infty} |p_i| = +\infty$ where the limit is taken for all i such that $p_i \in \mathbb{C}$;

or

(6.3.2.2) $p_0 \in \mathbb{C}$, all but a finite number of the p_i are in $\mathbb{C}$ and $\lim_{i \to \infty} p_i = p_0$ in the usual sense of convergence in $\mathbb{C}$.

6.3.3 The Riemann Sphere

Stereographic projection puts $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ into one-to-one correspondence with the two-dimensional sphere S in $\mathbb{R}^3, S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in such a way that topology is preserved in both directions of the correspondence.

In detail, begin by imagining the unit sphere bisected by the complex plane with the center of the sphere $(0, 0, 0)$ coinciding with the origin in the plane—see Figure 6.1. We define the stereographic projection as follows: If $P = (x, y) \in \mathbb{C}$, then connect P to the "north pole" N of the sphere with a line segment. The point $\pi(P)$ of intersection of this segment with the sphere is called the *stereographic projection* of P. Note that, under stereographic projection, the "point at infinity" in the plane corresponds to the north pole N of the sphere. For this reason, $\mathbb{C} \cup \{\infty\}$ is often thought of as "being" a sphere, and is then called, for historical reasons, the *Riemann sphere.*

The construction we have just described is another way to think about the "extended complex plane"—see §§6.3.2. In these terms, linear fractional transformations become homeomorphisms of $\mathbb{C} \cup \{\infty\}$ to itself. (Recall that a *homeomorphism* is, by definition, a one-to-one, onto, continuous mapping with a continuous inverse.)

> If $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a linear fractional transformation, then f is a one-to-one, onto, continuous function. Also, $f^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a linear fractional transformation, and is thus a one-to-one, onto, continuous function.

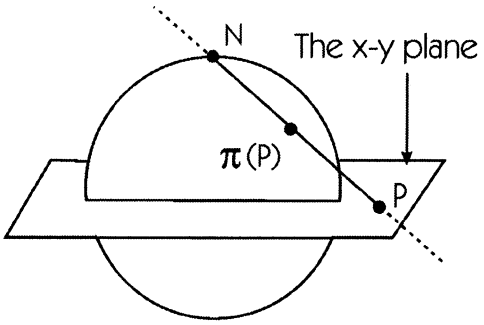

FIGURE 6.1
Stereographic projection.

If $g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is also a linear fractional transformation, then $f \circ g$ is a linear fractional transformation.

The simplicity of language obtained by adjoining ∞ to $\mathbb{C}$ (so that the composition and inverse properties of linear fractional transformations obviously hold) is well worth the trouble. Certainly one does not wish to consider the multiplicity of special possibilities when composing $(Az + B)/(Cz + D)$ with $(az + b)/(cz + d)$ (namely $c = 0, c \neq 0, aC + cD \neq 0, aC + cD = 0$, etc.) that arise every time composition is considered.

In fact, it is worth summarizing what we have learned in a theorem (see §§6.3.4). First note that it makes sense now to talk about a homeomorphism from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$ being conformal: this just means that it (and hence its inverse) are holomorphic in our extended sense. If φ is a conformal map of $\mathbb{C} \cup \{\infty\}$ to itself, then, after composing with a linear fractional transformation, we may suppose that φ maps ∞ to itself. Thus φ, after composition with a linear fraction transformation, is linear. It follows that φ itself is linear fractional. The following result summarizes the situation:

6.3.4 Conformal Self-Maps of the Riemann Sphere

A function φ is a conformal self-mapping of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself if and only if φ is linear fractional.

We turn now to the actual utility of linear fractional transformations (beyond their having been the form of automorphisms of D—see §§6.2.1–6.2.3—and the form of all conformal self maps of $\mathbb{C} \cup \{\infty\}$ to itself in the present section). One of the most frequently occurring uses is the following:

6.3.5 The Cayley Transform

The Cayley Transform The linear fractional transformation $z \mapsto (i - z)/(i + z)$ maps the upper half plane $\{z : \text{Im} z > 0\}$ conformally onto the unit disc $D = \{z : |z| < 1\}$.

6.3.6 Generalized Circles and Lines

Calculations of the type that we have been discussing are straightforward but tedious. It is thus worthwhile to seek a simpler way to understand what the image under a linear fractional transformation of a given region is. For regions bounded by line segments and arcs of circles the following result gives a method for addressing this issue:

Let $\mathcal{C}$ be the set of subsets of $\mathbb{C} \cup \{\infty\}$ consisting of (i) circles and (ii) sets of the form $L \cup \{\infty\}$ where L is a line in $\mathbb{C}$. We call the elements of $\mathcal{C}$ "generalized circles." Then every linear fractional transformation φ takes elements of $\mathcal{C}$ to elements of $\mathcal{C}$. One verifies this last assertion by noting that any linear fractional transformation is the composition of dilations, translations, and the inversion map $z \mapsto 1/z$; and each of these component maps clearly sends generalized circles to generalized circles.

6.3.7 The Cayley Transform Revisited

To illustrate the utility of this last result, we return to the Cayley transformation

$$z \mapsto \frac{i - z}{i + z}. \tag{6.3.7.1}$$

Under this mapping the point ∞ is sent to -1, the point 1 is sent to $(i - 1)/(i + 1) = i$, and the point -1 is sent to $(i - (-1))/(i + (-1)) = -i$. Thus the image under the Cayley transform (a linear fractional transformation) of three points on $\mathbb{R} \cup \{\infty\}$ contains three points on the unit circle. Since three points determine a (generalized) circle, and since linear fractional transformations send generalized circles to generalized circles, we may conclude that the Cayley transform sends the real line to the unit circle. Now the Cayley transform is one-to-one and onto from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$. By continuity, it either sends the upper half plane to the (open) unit disc or to the complement of the closed unit disc. The image of i is 0, so in fact the Cayley transform sends the upper half plane to the unit disc.

6.3.8 Summary Chart of Linear Fractional Transformations

The next chart summarizes the properties of some important linear fractional transformations. Note that $U = \{z \in \mathbb{C} : \text{Im} z > 0\}$ is the upper

half-plane and $D = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc; the domain variable is z and the range variable is w.

Linear Fractional Transformations

Domain	Image	Conditions	Formula
$z \in \widehat{\mathbb{C}}$	$w \in \widehat{\mathbb{C}}$		$w = \frac{az+b}{cz+d}$
$z \in D$	$w \in U$		$w = i \cdot \frac{1-z}{1+z}$
$z \in U$	$w \in D$		$w = \frac{i-z}{i+z}$
$z \in D$	$w \in D$		$w = \frac{z-a}{1-\overline{a}z}$, $\lvert a\rvert < 1$
$\mathbb{C}$	$\mathbb{C}$	$L(z_1) = w_1$	$L(z) = S^{-1} \circ T$
		$L(z_2) = w_2$	$T(z) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$
		$L(z_3) = w_3$	$S(m) = \frac{m-w_1}{m-w_3} \cdot \frac{w_2-w_3}{w_2-w_1}$

6.4 The Riemann Mapping Theorem

6.4.1 The Concept of Homeomorphism

Two open sets U and V in $\mathbb{C}$ are *homeomorphic* if there is a one-to-one, onto, continuous function $f : U \to V$ with $f^{-1} : V \to U$ also continuous. Such a function f is called a *homeomorphism* from U to V (see also §§6.3.3).

6.4.2 The Riemann Mapping Theorem

The Riemann mapping theorem, sometimes called the greatest theorem of the nineteenth century, asserts in effect that any planar domain that has the topology of the unit disc also has the conformal structure of the unit disc. Even though this theorem has been subsumed by the great uniformization theorem of Köbe (see [FAK]), it is still striking in its elegance and simplicity:

> If U is an open subset of $\mathbb{C}, U \neq \mathbb{C}$, and if U is homeomorphic to D, then U is conformally equivalent to D. That is, there is a holomorphic mapping $\psi : U \to D$ which is one-to-one and onto.

6.4.3 The Riemann Mapping Theorem: Second Formulation

An alternative formulation of this theorem uses the concept of "simply connected" (see also §§2.3.3.). We say that a connected open set U in the complex plane is simply connected if any closed curve in U can be continuously deformed to a point. (This is just a precise way of saying that U has no holes. Yet another formulation of the notion is that the complement of U has only one connected component.) See §§10.3.2 for a detailed treatment of homotopy (which is the rigorous notion of "continuous deformation").

> **Theorem:** If U is an open subset of $\mathbb{C}, U \neq \mathbb{C}$, and if U is simply connected, then U is conformally equivalent to D.

6.5 Conformal Mappings of Annuli

6.5.1 A Riemann Mapping Theorem for Annuli

The Riemann mapping theorem tells us that, from the point of view of complex analysis, there are only two simply connected planar domains: the disc and the plane. Any other simply connected region is biholomorphic to one of these. It is natural then to ask about domains with holes. Take, for example, a domain U with precisely one hole. Is it conformally equivalent to an annulus?

Note that if $c > 0$ is a constant, then for any $R_1 < R_2$ the annuli

$$A_1 \equiv \{z : R_1 < |z| < R_2\} \text{ and } A_2 \equiv \{z : cR_1 < |z| < cR_2\} \quad (6.5.1.1)$$

are biholomorphically equivalent under the mapping $z \mapsto cz$. The surprising fact that we shall learn is that these are the *only* circumstances under which two annuli are equivalent:

6.5.2 Conformal Equivalence of Annuli

Let

$$A_1 = \{z \in \mathbb{C} : 1 < |z| < R_1\} \quad (6.5.2.1)$$

and

$$A_2 = \{z \in \mathbb{C} : 1 < |z| < R_2\}. \quad (6.5.2.2)$$

Then A_1 is conformally equivalent to A_2 if and only if $R_1 = R_2$.

A perhaps more striking result, and more difficult to prove, is this:

> Let $U \subseteq \mathbb{C}$ be any bounded domain with *one hole*—this means that the complement of U has two connected components, one bounded and one not. Then U is conformally equivalent to some annulus.

See [AHL] for a discursive discussion of this result.

6.5.3 Classification of Planar Domains

The classification of planar domains up to biholomorphic equivalence is a part of the theory of Riemann surfaces (see §10.4). For now, we comment that one of the startling classification theorems (a generalization of the Riemann mapping theorem) is that any bounded planar domain with finitely many "holes" is conformally equivalent to the unit disc with finitely many closed circular arcs, coming from circles centered at the origin, removed. (Here a "hole" in the present context means a bounded, connected component of the complement of the domain in $\mathbb{C}$, a concept which coincides with the intuitive idea of a hole.) An alternative equivalent statement is that any bounded planar domain with finitely many holes is conformally equivalent to the plane with finitely many vertical slits (see [AHL]). The analogous result for domains with infinitely many holes is known to be true when the number of holes is countable.

Chapter 7

Harmonic Functions

7.1 Basic Properties of Harmonic Functions

7.1.1 The Laplace Equation

Let F be a holomorphic function on an open set $U \subseteq \mathbb{C}$. Write $F = u + iv$, where u and v are real-valued. The real part u satisfies a certain partial differential equation known as *Laplace's equation*:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0. \tag{7.1.1.1}$$

(Of course the imaginary part v satisfies the same equation.) In this chapter we shall study systematically those C^2 functions that satisfy this equation. They are called *harmonic* functions. (Note that we encountered some of these ideas already in §1.4.)

7.1.2 Definition of Harmonic Function

Recall the precise definition of harmonic function:

A real-valued function $u : U \to \mathbb{R}$ on an open set $U \subseteq \mathbb{C}$ is *harmonic* if it is C^2 on U and

$$\Delta u \equiv 0, \tag{7.1.2.1}$$

where the Laplacian Δu is defined by

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u. \tag{7.1.2.2}$$

7.1.3 Real- and Complex-Valued Harmonic Functions

The definition of harmonic function just given applies as well to complex-valued functions. A complex-valued function is harmonic if and only if its real and imaginary parts are each harmonic.

S. G. Krantz, *Handbook of Complex Variables*

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The first thing that we need to check is that real-valued harmonic functions are just those functions that arise as the real parts of holomorphic functions—at least locally.

7.1.4 Harmonic Functions as the Real Parts of Holomorphic Functions

If $u : D(P,r) \to \mathbb{R}$ is a harmonic function on a disc $D(P,r)$, then there is a holomorphic function $F : D(P,r) \to \mathbb{C}$ such that $\operatorname{Re} F \equiv u$ on $D(P,r)$.

Note that v is uniquely determined by u except for an additive constant: the Cauchy-Riemann equations determine the partial derivatives of v and hence determine v up to an additive constant. One can also think of the determination, up to a constant, of v by u in another way: If $\widetilde{v}$ is another function such that $u + i\widetilde{v}$ is holomorphic, then $H \equiv i(v - \widetilde{v}) = (u + iv) - (u + i\widetilde{v})$ is a holomorphic function with zero real part; hence its image is not open. Thus H must be a constant, and v and $\widetilde{v}$ differ by a constant. Any (harmonic) function v such that $u + iv$ is holomorphic is called a *harmonic conjugate* of u (again see §§1.4.2).

> **Theorem:** If U is a simply connected open set (see §§6.4.3) and if $u : U \to \mathbb{R}$ is a harmonic function, then there is a C^2 (indeed a C^∞) harmonic function v such that $u + iv : U \to \mathbb{C}$ is holomorphic.

Another important relationship between harmonic and holomorphic functions is this:

> If $u : U \to \mathbb{R}$ is harmonic and if $H : V \to U$ is holomorphic, then $u \circ H$ is harmonic on V.

7.1.5 Smoothness of Harmonic Functions

If $u : U \to \mathbb{R}$ is a harmonic function on an open set $U \subseteq \mathbb{C}$, then $u \in C^\infty$. In fact a harmonic function is always real analytic (has a local power series expansion in powers of x and y). This follows, for instance, because a harmonic function is locally the real part of a holomorphic function (see §§1.4.2, §§7.1.4).

7.2 The Maximum Principle and the Mean Value Property

7.2.1 The Maximum Principle for Harmonic Functions

If $u : U \to \mathbb{R}$ is harmonic on a connected open set U and if there is a point $P_0 \in U$ with the property that $u(P_0) = \max_{z \in U} u(z)$, then u is constant on U. Compare the maximum modulus principle for holomorphic functions in §§5.4.1.

7.2.2 The Minimum Principle for Harmonic Functions

> If $u : U \to \mathbb{R}$ is a harmonic function on a connected open set $U \subseteq \mathbb{C}$ and if there is a point $P_0 \in U$ such that $u(P_0) = \min_{Q \in U} u(Q)$, then u is constant on U. Compare the minimum principle for holomorphic functions in §§5.4.3.

The reader may note that the minimum principle for holomorphic functions requires an extra hypothesis (i.e., non-vanishing of the function) while that for harmonic functions does not. The difference may be explained by noting that with harmonic functions we are considering the real-valued function u, while with holomorphic functions we must restrict attention to the modulus function $|f|$.

7.2.3 The Boundary Maximum and Minimum Principles

An important and intuitively appealing consequence of the maximum principle is the following result (which is sometimes called the "boundary maximum principle"). Recall that a continuous function on a compact set assumes a maximum value. When the function is harmonic, the maximum occurs at the boundary in the following precise sense:

> Let $U \subseteq \mathbb{C}$ be a bounded domain. Let u be a continuous, real-valued function on the closure $\overline{U}$ of U that is harmonic on U. Then
>
> $$\max_{\overline{U}} u = \max_{\partial U} u. \tag{7.2.3.1}$$

The analogous result for the minimum is:

> Let $U \subseteq \mathbb{C}$ be a domain and let u be a continuous function on the closure $\overline{U}$ of U that is harmonic on U. Then
>
> $$\min_{\overline{U}} u = \min_{\partial U} u. \tag{7.2.3.2}$$

Compare the analogous results for holomorphic functions in §§5.4.2, 5.4.3.

7.2.4 The Mean Value Property

Suppose that $u : U \to \mathbb{R}$ is a harmonic function on an open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P,r) \subseteq U$ for some $r > 0$. Then

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta})\, d\theta. \tag{7.2.4.1}$$

We will see in §§7.4.1 that the mean value property characterizes harmonic functions.

We conclude this subsection with two alternative formulations of the mean value property (MVP). In both, u, U, P, r are as above.

First Alternative Formulation of MVP

$$u(P) = \frac{1}{\pi r^2} \iint_{D(P,r)} u(x,y)\, dxdy.$$

Second Alternative Formulation of MVP

$$u(P) = \frac{1}{2\pi r} \int_{\partial D(P,r)} u(\zeta)\, d\sigma(\zeta),$$

where $d\sigma$ is arc-length measure on $\partial D(P,r)$.

7.2.5 Boundary Uniqueness for Harmonic Functions

If $u_1 : \overline{D}(0,1) \to \mathbb{R}$ and $u_2 : \overline{D}(0,1) \to \mathbb{R}$ are two continuous functions, each of which is harmonic on $D(0,1)$ and if $u_1 = u_2$ on $\partial D(0,1) = \{z : |z| = 1\}$, then $u_1 \equiv u_2$. This assertion follows from the boundary maximum principle (§§7.2.3.1) applied to $u_1 - u_2$. Thus, in effect, a harmonic function u on $D(0,1)$ that extends continuously to $\overline{D}(0,1)$ is completely determined by its values on $\overline{D}(0,1) \setminus D(0,1) = \partial D(0,1)$.

7.3 The Poisson Integral Formula

7.3.1 The Poisson Integral

The next result shows how to calculate a harmonic function on the disc from its "boundary values," i.e., its values on the circle that bounds the disc.

Let $u : U \to \mathbb{R}$ be a harmonic function on a neighborhood of $\overline{D}(0,1)$. Then, for any point $a \in D(0,1)$,

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\psi}) \cdot \frac{1 - |a|^2}{|a - e^{i\psi}|^2} \, d\psi. \tag{7.3.1.1}$$

7.3.2 The Poisson Kernel

The expression

$$\frac{1}{2\pi} \frac{1 - |a|^2}{|a - e^{i\psi}|^2} \tag{7.3.2.1}$$

is called the *Poisson kernel* for the unit disc. It is often convenient to rewrite the formula we have just enunciated by setting $a = |a|e^{i\theta} = re^{i\theta}$. Then the result says that

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\psi}) \frac{1 - r^2}{1 - 2r\cos(\theta - \psi) + r^2} \, d\psi. \tag{7.3.2.2}$$

In other words

$$u(re^{i\theta}) = \int_0^{2\pi} u(e^{i\theta}) P_r(\theta - \psi) \, d\psi, \tag{7.3.2.3}$$

where

$$P_r(\theta - \psi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \psi) + r^2}. \tag{7.3.2.4}$$

7.3.3 The Dirichlet Problem

The Poisson integral formula both reproduces and creates harmonic functions. But, in contrast to the holomorphic case (§2.4), there is a simple connection between a continuous function f on $\partial D(0,1)$ and the created harmonic function u on D. The following theorem states this connection precisely. The theorem is usually called "the solution of the Dirichlet problem on the disc":

7.3.4 The Solution of the Dirichlet Problem on the Disc

Let f be a continuous function on $\partial D(0,1)$. Define

$$u(z) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(e^{i\psi}) \cdot \frac{1 - |z|^2}{|z - e^{i\psi}|^2} \, d\psi & \text{if } z \in D(0,1) \\ f(z) & \text{if } z \in \partial D(0,1). \end{cases} \tag{7.3.4.1}$$

Then u is continuous on $\overline{D}(0,1)$ and harmonic on $D(0,1)$.

Closely related to this result is the *reproducing property* of the Poisson kernel:

Let u be harmonic on a neighborhood of $\overline{D}(0,1)$. Then, for $z \in D(0,1)$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi}) \cdot \frac{1-|z|^2}{|z-e^{i\psi}|^2}\, d\psi.$$

See (7.3.1.1).

7.3.5 The Dirichlet Problem on a General Disc

A change of variables shows that the results of §§7.3.4 remain true on a general disc. To wit, let f be a continuous function on $\partial D(P,r)$. Define

$$u(z) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\psi}) \cdot \frac{R-|z-P|^2}{|(z-P)-Re^{i\psi}|^2}\, d\psi & \text{if } z \in D(0,1) \\ f(z) & \text{if } z \in \partial D(0,1). \end{cases} \tag{7.3.5.1}$$

Then u is continuous on $\overline{D}(0,1)$ and harmonic on $D(0,1)$.

If instead u is harmonic on a neighborhood of $\overline{D}(P,R)$, then, for $z \in D(P,R)$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(P+Re^{i\psi}) \cdot \frac{R^2-|z-P|^2}{|(z-P)-Re^{i\psi}|^2}\, d\psi. \tag{7.3.5.2}$$

7.4 Regularity of Harmonic Functions

7.4.1 The Mean Value Property on Circles

A continuous function $h : U \to \mathbb{R}$ on an open set $U \subseteq \mathbb{C}$ has the *ϵ_P-mean value property* if, for each point $P \in U$, there is an $\epsilon_P > 0$ such that $\overline{D}(P,\epsilon_P) \subseteq U$ and, for every $0 < \epsilon < \epsilon_P$,

$$h(P) = \frac{1}{2\pi}\int_0^{2\pi} h(P+\epsilon e^{i\theta})\, d\theta. \tag{7.4.1.1}$$

The ϵ_P-mean value property allows the size of ϵ_P to vary arbitrarily with P.

> If $h : U \to \mathbb{R}$ is a continuous function on an open set U with the ϵ_P-mean value property, then h is harmonic.

7.4.2 The Limit of a Sequence of Harmonic Functions

If $\{h_j\}$ is a sequence of real-valued harmonic functions that converges uniformly on compact subsets of U to a function $h : U \to \mathbb{R}$, then h is harmonic on U.

7.5 The Schwarz Reflection Principle

7.5.1 Reflection of Harmonic Functions

We present in this section an application of what we have learned so far to a question of extension of a harmonic function to a larger domain. This will already illustrate the importance and power of harmonic function theory and will provide us with a striking result about holomorphic functions as well.

7.5.2 Schwarz Reflection Principle for Harmonic Functions

Let V be a connected open set in $\mathbb{C}$. Suppose that

$$V \cap (\text{real axis}) = \{x \in \mathbb{R} : a < x < b\}.$$

Set $U = \{z \in V : \operatorname{Im} z > 0\}$. Assume that $v : U \to \mathbb{R}$ is harmonic and that, for each $\zeta \in V \cap (\text{real axis})$,

$$\lim_{U \ni z \to \zeta} v(z) = 0. \tag{7.5.2.1}$$

Set $\widetilde{U} = \{\overline{z} : z \in U\}$. Define

$$\widehat{v}(z) = \begin{cases} v(z) & \text{if } z \in U \\ 0 & \text{if } z \in V \cap (\text{real axis}) \\ -v(\overline{z}) & \text{if } z \in \widetilde{U}. \end{cases} \tag{7.5.2.2}$$

Then $\widehat{v}$ is harmonic on $U^* \equiv U \cup \widetilde{U} \cup \{x \in \mathbb{R} : a < x < b\}$.

This result provides a way of extending a harmonic function from a given open set to a larger open set. The method is known as the *Schwarz Reflection Principle*. One can think of $\widehat{U}$ as the reflection of U in the real axis, and the definition of $\widehat{v}$ on $\widehat{U}$ as the correspondingly appropriate idea of reflecting the function v. See Figure 7.1.

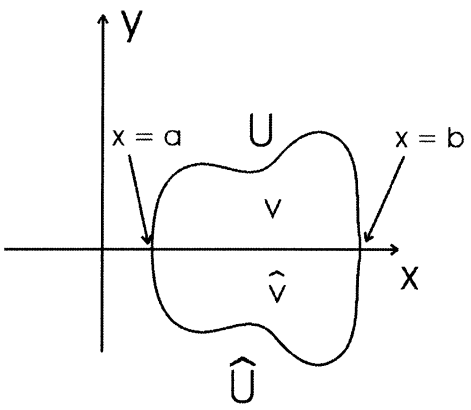

FIGURE 7.1
Schwarz reflection.

7.5.3 The Schwarz Reflection Principle for Holomorphic Functions

Let V be a connected open set in $\mathbb{C}$ such that $V \cap$ (the real axis) $= \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$. Set $U = \{z \in V : \operatorname{Im} z > 0\}$. Suppose that $F : U \to \mathbb{C}$ is holomorphic and that

$$\lim_{U \ni z \to x} \operatorname{Im} F(z) = 0 \tag{7.5.3.1}$$

for each $\zeta \in V \cap$ (real axis). Define $\widetilde{U} = \{z \in \mathbb{C} : \overline{z} \in U\}$. Then there is a holomorphic function G on $U \cup \widetilde{U} \cup \{x \in \mathbb{R} : a < x < b\}$ such that $G\big|_U = F$. In fact $\varphi(\zeta) \equiv \lim_{U \ni z \to \zeta} \operatorname{Re} F(z)$ exists for each $\zeta \in V \cap$ (real axis) and

$$G(z) = \begin{cases} F(z) & \text{if } z \in U \\ \varphi(z) + i0 & \text{if } z \in V \cap \text{ (real axis)} \\ \overline{F(\overline{z})} & \text{if } z \in \widetilde{U}. \end{cases} \tag{7.5.3.2}$$

7.5.4 More General Versions of the Schwarz Reflection Principle

We take this opportunity to note that Schwarz reflection is not simply a fact about reflection in lines. Since lines are conformally equivalent (by way of linear fractional transformations) to circles, it is also possible to perform Schwarz reflection in a circle (with suitably modified hypotheses). More is true: in fact one can conformally map a neighborhood of any real analytic curve to a line segment; so, with some extra effort, Schwarz reflection may be performed in any real analytic arc.

7.6 Harnack's Principle

7.6.1 The Harnack Inequality

Let u be a non-negative, harmonic function on $\overline{D}(0,R)$. Then, for any $z \in D(0,R)$,

$$\frac{R-|z|}{R+|z|} \cdot u(0) \leq u(z) \leq \frac{R+|z|}{R-|z|} \cdot u(0). \tag{7.6.1.1}$$

More generally:

Let u be a non-negative, harmonic function on $\overline{D}(P,R)$. Then, for any $z \in D(P,R)$,

$$\frac{R-|z-P|}{R+|z-P|} \cdot u(P) \leq u(z) \leq \frac{R+|z-P|}{R-|z-P|} \cdot u(P). \tag{7.6.1.2}$$

7.6.2 Harnack's Principle

Let $u_1 \leq u_2 \leq \ldots$ be harmonic functions on a connected open set $U \subseteq \mathbb{C}$. Then either $u_j \to \infty$ uniformly on compact sets or there is a (finite-valued) harmonic function u on U such that $u_j \to u$ uniformly on compact sets.

7.7 The Dirichlet Problem and Subharmonic Functions

7.7.1 The Dirichlet Problem

Let $U \subseteq \mathbb{C}$ be an open set, $U \neq \mathbb{C}$. Let f be a given continuous function on ∂U. Does there exist a continuous function u on $\overline{U}$ such that $u\big|_{\partial U} = f$ and u is harmonic on U? If u exists, is it unique? These two questions taken together are called the *Dirichlet problem* for the domain U. [Note that we have already solved the Dirichlet problem when U is the unit disc—see §§7.3.1, §§7.3.4.] It has many motivations from physics (see [COH], [LOG]). For instance, suppose that a flat, thin film of heat-conducting material is in thermal equilibrium. That is, the temperature at each point of the film is constant with passing time (§§14.2.2). Then its temperature at various points is a harmonic function (see [KRA4]). Physical intuition suggests that if the boundary ∂U of the film has a given temperature distribution f :

$\partial U \to \mathbb{R}$, then the temperatures at interior points are uniquely determined. Historically, physicists have found this intuition strongly compelling.

From the viewpoint of mathematical proof, as opposed to physical intuition, the situation is more complicated. The result of §§7.3.4 asserts in effect that the Dirichlet problem on the unit disc always has a solution. And, on any bounded domain U, it has only one solution corresponding to any given boundary function f, because of the (boundary) maximum principle: If u_1 and u_2 are both solutions, then $u_1 - u_2$ is harmonic and is zero on the boundary, so that $u_1 - u_2 \equiv 0$, hence $u_1 \equiv u_2$. While this reasoning demonstrates that the Dirichlet problem on a bounded open set U can have *at most* one solution, it is also the case that on more complicated domains the Dirichlet problem may not have *any* solution.

7.7.2 Conditions for Solving the Dirichlet Problem

The Dirichlet problem is not always solvable on the domain $U = D(0,1)\setminus\{0\}$; in fact the data $f(z) = 1$ when $|z| = 1$ and $f(z) = 0$ when $z = 0$ have no solution—see [GK, p. 229]. Thus *some* conditions on ∂U are necessary in order that the Dirichlet problem be solvable for U. It will turn out that if ∂U consists of "smooth" curves, then the Dirichlet problem is always solvable. The best possible general result is that if each connected component of the boundary of U contains more than one point, then the Dirichlet problem can always be solved. Later, we shall enunciate a classical condition for solvability of the Dirichlet formulated in the language of barriers.

7.7.3 Motivation for Subharmonic Functions

We first consider the concept of subharmonicity. This is a complex-analytic analogue of the notion of convexity that we motivate by considering convexity on the real line. For the moment, fix attention on functions $F : \mathbb{R} \to \mathbb{R}$.

On the real line, the analogue of the Laplacian is the operator d^2/dx^2. The analogue of real-valued harmonic functions (that is, the functions u with $\triangle u = 0$) are therefore the functions $h(x)$ such that $[d^2/dx^2][h(x)] \equiv 0$; these are the linear ones. Let $\mathcal{S}$ be the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that whenever $I = [a,b] \subseteq \mathbb{R}$ and h is a real-valued harmonic function on $\mathbb{R}$ with $f(a) \leq h(a)$ and $f(b) \leq h(b)$, then $f(x) \leq h(x)$ for all $x \in I$. (Put simply, if a harmonic function h is at least as large as f at the endpoints of an interval, then it is at least as large as f on the entire interval.) Which functions are in $\mathcal{S}$? The answer is the collection of all convex functions (in the usual sense). Refer to Figure 7.2. [Recall here that a function $f : [a,b] \to \mathbb{R}$ is said to be convex if, whenever $c, d \in [a,b]$ and $0 \leq \lambda \leq 1$ then $f((1-\lambda)c + \lambda d) \leq (1-\lambda)f(c) + \lambda f(d)$.] These considerations give us a geometric way to think about convex functions (without resorting to

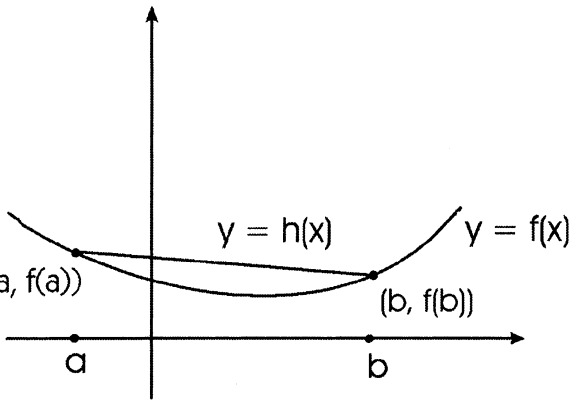

FIGURE 7.2
A convex function.

differentiation). See [HOR] for more on this view of subharmonic functions.

7.7.4 Definition of Subharmonic Function

Our definition of subharmonic function on a domain in $\mathbb{C}$ (or $\mathbb{R}^2$) is motivated by the discussion of convexity in the preceding subsection.

> Let $U \subseteq \mathbb{C}$ be an open set and f a real-valued, continuous function on U. Suppose that for each $\overline{D}(P,r) \subseteq U$ and every real-valued harmonic function h defined on a neighborhood of $\overline{D}(P,r)$ which satisfies $f \leq h$ on $\partial D(P,r)$, it holds that $f \leq h$ on $D(P,r)$. Then f is said to be *subharmonic* on U.

7.7.5 Other Characterizations of Subharmonic Functions

A function $f : U \to \mathbb{R}$ that is C^2 is subharmonic if and only if $\triangle f \geq 0$ everywhere. This is analogous to the fact that a C^2 function on (an open set in) $\mathbb{R}$ is convex if and only if it has non-negative second derivative everywhere. The next result will allow us to identify many subharmonic functions which are only continuous, not C^2, so that the $\triangle f \geq 0$ criterion is not applicable.

Let $f : U \to \mathbb{R}$ be continuous. Suppose that, for each $\overline{D}(P,r) \subseteq U$,

$$f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) d\theta \tag{7.7.5.1}$$

(this is called the *sub-mean value property*). Then f is subharmonic.

Conversely, if $f : U \to \mathbb{R}$ is a continuous, subharmonic function and if $\overline{D}(P,r) \subseteq U$, then the inequality (7.7.5.1) holds.

7.7.6 The Maximum Principle

A consequence of the sub-mean value property (7.7.5.1) is the maximum principle for subharmonic functions:

> If f is subharmonic on U and if there is a $P \in U$ such that $f(P) \geq f(z)$ for all $z \in U$, then f is constant.

It may be noted that if f is holomorphic then $|f|$ is subharmonic; this explains why a holomorphic function satisfies the maximum principle.

7.7.7 Lack of A Minimum Principle

We note in passing that there is no "minimum principle" for subharmonic functions. Subharmonicity is a "one-sided" property.

7.7.8 Basic Properties of Subharmonic Functions

Here are some properties of subharmonic functions that are worth noting. The third of these explains why subharmonic functions are a much more flexible tool than holomorphic or even harmonic functions.

1. If f_1, f_2 are subharmonic functions on U, then so is $f_1 + f_2$.
2. If f_1 is subharmonic on U and $\alpha > 0$ is a constant, then αf_1 is subharmonic on U.
3. If f_1, f_2 are subharmonic on U then $g(z) \equiv \max\{f_1(z), f_2(z)\}$ is also subharmonic on U.

7.7.9 The Concept of a Barrier

The next notion that we need to introduce is that of a *barrier*. Namely, we want to put a geometric-analytic condition on the boundary of a domain that will rule out examples like the punctured disc in §§7.7.2 (in which the Dirichlet problem could not be solved). The definition of a barrier at a point $P \in \partial U$ is a bit technical, but the existence of a barrier will turn out to be exactly the hypothesis needed for the construction of the solution of the Dirichlet problem.

> Let $U \subseteq \mathbb{C}$ be an open set and $P \in \partial U$. We call a function $b : \overline{U} \to \mathbb{R}$ a *barrier* for U at P if

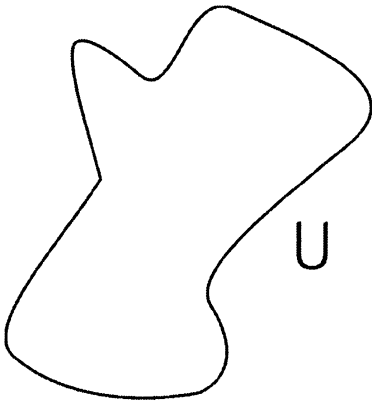

FIGURE 7.3
Smooth boundary means a barrier at each point of the boundary.

(7.7.9.1) b is continuous;

(7.7.9.2) b is subharmonic on U;

(7.7.9.3) $b\big|_{\partial U} \leq 0$;

(7.7.9.4) $\{z \in \partial U : b(z) = 0\} = \{P\}$.

Thus the barrier b singles out P in a special function-theoretic fashion.

If U is bounded by a C^1 smooth curve (no corners present), then every point of ∂U has a barrier. See Figure 7.3.

7.8 The General Solution of the Dirichlet Problem

7.8.1 Enunciation of the Solution of the Dirichlet Problem

Let U be a bounded, connected open subset of $\mathbb{C}$ such that U has a barrier b_P for each $P \in \partial U$. Then the Dirichlet problem can always be solved on U. That is, if f is a continuous function on ∂U, then there is a function u continuous on $\overline{U}$, harmonic on U, such that $u\big|_{\partial U} = f$. The function u is uniquely determined by these conditions.

The result in the preceding paragraph is the standard textbook result about regularity for the Dirichlet problem. More advanced techniques establish that if each connected component of ∂U has at least two points then the Dirichlet problem is solvable on U.

Chapter 8

Infinite Series and Products

8.1 Basic Concepts Concerning Infinite Sums and Products

8.1.1 Uniform Convergence of a Sequence

Let $U \subseteq \mathbb{C}$ be an open set and $g_j : U \to \mathbb{C}$ functions. Recall (§3.1.5) that the sequence $\{g_j\}$ is said to *converge uniformly on compact subsets of* U to a function g if the following condition holds: For each compact $K \subseteq U$ (see §§3.1.5) and each $\epsilon > 0$ there is an $N > 0$ such that if $j > N$, then $|g(z) - g_j(z)| < \epsilon$ for all $z \in K$. It should be noted that, in general, the choice of N depends on ϵ *and* on K, but not on the particular point $z \in K$.

8.1.2 The Cauchy Condition for a Sequence of Functions

Because of the completeness (see [ℝUD1], [KRA2]) of the complex numbers, a sequence is uniformly convergent on compact sets if and only if it is (what is called) *uniformly Cauchy* on compact sets. Here a sequence of functions is said to be *uniformly Cauchy on compact sets* if, for each K compact in U and each $\epsilon > 0$, there is an $N > 0$ such that: for all $j, k > N$ and all $z \in K$ we have $|g_j(z) - g_k(z)| < \epsilon$. The Cauchy condition is useful because it does not make explicit reference to the limit function g. One can thereby verify uniform convergence to some limit without previously determining the limit function explicitly.

8.1.3 Normal Convergence of a Sequence

For the purposes of complex analysis, the basic fact about uniform convergence on compact sets is our result which states that if $\{g_j\}$ are holomorphic functions on U and if the g_j converge uniformly on compact subsets of U to a function g, then g is holomorphic (§§3.1.5). In these circumstances we will say that $\{g_j\}$ converges *normally* to g.

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8.1.4 Normal Convergence of a Series

If $f_1, f_2, \ldots$ are functions on U, then we may study the convergence properties of

$$\sum_{j=1}^{\infty} f_j. \tag{8.1.4.1}$$

The series converges *normally* if its sequence of partial sums

$$S_N(z) = \sum_{j=1}^{N} f_j(z) \ , \qquad N = 1, 2, \ldots \tag{8.1.4.2}$$

converges normally in U. The function

$$f(z) = \sum_{j=1}^{\infty} f_j(z) \tag{8.1.4.3}$$

will then be holomorphic because it is the normal limit of the partial sums S_N (each of which is holomorphic).

8.1.5 The Cauchy Condition for a Series

There is a Cauchy condition for normal convergence of a series: the series

$$\sum_{j=1}^{\infty} f_j(z) \tag{8.1.5.1}$$

is said to be *uniformly Cauchy* on compact sets if, for each compact $K \subseteq U$ and each $\epsilon > 0$, there is an $N > 0$ such that for all $M \geq L > N$ it holds that

$$\left| \sum_{j=L}^{M} f_j(z) \right| < \epsilon. \tag{8.1.5.2}$$

[Note that this is just a reformulation of the Cauchy condition for the sequence of partial sums $S_N(z)$.] It is easy to see that a series that is uniformly Cauchy converges normally to its limit function.

8.1.6 The Concept of an Infinite Product

Now we turn to products. One of the principal activities in complex analysis is to construct holomorphic or meromorphic functions with certain prescribed behavior. For some problems of this type, it frequently turns out that infinite products are more useful than infinite sums. The reason is that, for instance, if we want to construct a function that will vanish on a

certain infinite set $\{a_j\}$, then we could hope to find individual functions f_j that vanish at a_j and then multiply the f_j's together. This process requires that we make sense of the notion of "infinite product."

8.1.7 Infinite Products of Scalars

We begin with infinite products of complex numbers, and then adapt the ideas to infinite products of functions. For reasons that will become apparent momentarily, it is convenient to write products in the form

$$\prod_{j=1}^{\infty}(1+a_j), \tag{8.1.7.1}$$

where $a_j \in \mathbb{C}$. The symbol $\prod$ stands for multiplication. We want to define what it means for a product such as (8.1.7.1) to converge.

8.1.8 Partial Products

First define the *partial products* P_N of (8.1.7.1) to be

$$P_N = \prod_{j=1}^{N}(1+a_j) \equiv (1+a_1)\cdot(1+a_2)\cdots(1+a_N). \tag{8.1.8.1}$$

We might be tempted to say that the infinite product

$$\prod_{j=1}^{\infty}(1+a_j) \tag{8.1.8.2}$$

converges if the sequence of partial products $\{P_N\}$ converges. But, for technical reasons, a different definition is more useful.

8.1.9 Convergence of an Infinite Product

An infinite product

$$\prod_{j=1}^{\infty}(1+a_j) \tag{8.1.9.1}$$

is said to *converge* if

(8.1.8.2) Only a finite number $a_{j_1}, \ldots, a_{j_k}$ of the a_j's are equal to -1.

(8.1.9.3) If $N_0 > 0$ is so large that $a_j \neq -1$ for $j > N_0$, then

$$\lim_{N\to+\infty} \prod_{j=N_0+1}^{N} (1+a_j) \tag{8.1.9.3.1}$$

exists and is non-zero.

8.1.10 The Value of an Infinite Product

If $\prod_{j=1}^{\infty}(1+a_j)$ converges, then we define its *value* to be (with N_0 as in (8.1.9.3))

$$\left[\prod_{j=1}^{N_0}(1+a_j)\right]\cdot\lim_{N\to+\infty}\prod_{N_0+1}^{N}(1+a_j). \tag{8.1.10.1}$$

8.1.11 Products That Are Disallowed

As the exposition develops, it will become clear why we wish to disallow products with

$$\lim_{N\to+\infty}\prod_{j=N_0+1}^{N}(1+a_j)=0. \tag{8.1.11.1}$$

8.1.12 Condition for Convergence of an Infinite Product

If

$$\sum_{j=1}^{\infty}|a_j|<\infty\,, \tag{8.1.12.1}$$

then

$$\prod_{j=1}^{\infty}(1+|a_j|) \tag{8.1.12.2}$$

converges. [See (8.1.12.6), (8.1.12.7) for part of the mathematical reason as to why these assertions are true.]

If

$$\prod_{j=1}^{\infty}(1+|a_j|) \tag{8.1.12.3}$$

converges, then

$$\sum_{j=1}^{\infty}|a_j| \tag{8.1.12.4}$$

converges.

Let $a_j\in\mathbb{C}$. Set

$$P_N=\prod_{j=1}^{N}(1+a_j)\,,\qquad \widetilde{P}_N=\prod_{j=1}^{N}(1+|a_j|). \tag{8.1.12.5}$$

Then

(8.1.12.6) $\widetilde{P}_N \leq \exp(|a_1| + \cdots + |a_N|)$.

(8.1.12.7) $|P_N - 1| \leq \widetilde{P}_N - 1$.

If the infinite product

$$\prod_{j=1}^{\infty}(1 + |a_j|) \tag{8.1.12.8}$$

converges, then so does

$$\prod_{j=1}^{\infty}(1 + a_j). \tag{8.1.12.9}$$

If

$$\sum_{j=1}^{\infty}|a_j| < \infty, \tag{8.1.12.10}$$

then

$$\prod_{j=1}^{\infty}(1 + a_j) \tag{8.1.12.11}$$

converges.

This last is our most useful convergence result for infinite products. It is so important that it is worth restating in a standard alternative form: If

$$\sum_{j=1}^{\infty}|1 - a_j| < \infty, \tag{8.1.12.12}$$

then

$$\prod_{j=1}^{\infty}a_j \tag{8.1.12.13}$$

converges. The proof is just a change of notation.

8.1.13 Infinite Products of Holomorphic Functions

We now apply these considerations to infinite products of holomorphic functions.

Let $U \subseteq \mathbb{C}$ be open. Suppose that $f_j : U \to \mathbb{C}$ are holomorphic and that

$$\sum_{j=1}^{\infty}|f_j| \tag{8.1.13.1}$$

converges uniformly on compact subsets of U. Then the sequence of partial products

$$F_N(z) = \prod_{j=1}^{N}(1 + f_j(z)) \tag{8.1.13.2}$$

converges uniformly on compact subsets of U. In particular, the limit of these partial products defines a holomorphic function on U.

8.1.14 Vanishing of an Infinite Product

The function f defined on a domain U by the product

$$f(z) = \prod_{j=1}^{\infty}(1 + f_j(z)) \tag{8.1.14.1}$$

vanishes at a point $z_0 \in U$ if and only if $f_j(z_0) = -1$ for some j. The multiplicity of the zero at z_0 is the sum of the multiplicities of the zeros of the functions $1 + f_j$ at z_0.

8.1.15 Uniform Convergence of an Infinite Product of Functions

Remark: For convenience, one says that the product $\prod_1^{\infty}(1 + f_j(z))$ converges uniformly on a set E if

(8.1.15.1) it converges for each z in E

and

(8.1.15.2) the sequence $\{\prod_1^N(1 + f_j(z))\}$ converges uniformly on E to $\prod_1^{\infty}(1 + f_j(z))$.

Then our main convergence result can be summarized as follows:

8.1.16 Condition for the Uniform Convergence of an Infinite Product of Functions

Theorem: If $\sum_{j=1}^{\infty}|f_j|$ converges uniformly on compact sets, then the product $\prod_1^{\infty}(1 + f_j(z))$ converges uniformly on compact sets.

It should be noted that the convergence conditions in §§8.1.9 are satisfied automatically in the situation of this last theorem.

8.2 The Weierstrass Factorization Theorem

8.2.1 Prologue

One of the most significant facts about a polynomial function $p(z)$ of $z \in \mathbb{C}$ is that it can be factored (see §§3.1.4):

$$p(z) = c \cdot \prod_{j=1}^{k} (z - a_j). \tag{8.2.1.1}$$

Among other things, such a factorization facilitates the study of the zeros of p. In this section we shall show that in fact any entire function can be factored in such a way that each multiplicative factor possesses precisely one zero (of first order). Since a function holomorphic on all of $\mathbb{C}$ (called an *entire function*) can have infinitely many zeros, the factorization must be an infinite product in at least some cases. We consider such a factorization in this section.

8.2.2 Weierstrass Factors

To obtain the Weierstrass factors, we define

$$E_0(z) = 1 - z \tag{8.2.2.1}$$

and for $1 \leq p \in \mathbb{Z}$ we let

$$E_p(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right). \tag{8.2.2.2}$$

Of course each E_p is holomorphic on all of $\mathbb{C}$. The factorization theory hinges on a technical calculation that says that, in some sense, E_p is close to 1 if $|z|$ is small. This assertion is not surprising since

$$\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \tag{8.2.2.3}$$

is the initial part of the power series of $-\log(1 - z)$. Thus

$$(1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \tag{8.2.2.4}$$

might be expected to be close to 1 for z small (and p large).

8.2.3 Convergence of the Weierstrass Product

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers with no accumulation point in the complex plane (note, however, that the a_n's need not be distinct). If $\{p_n\}$ are positive integers that satisfy

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \tag{8.2.3.1}$$

for every $r > 0$, then the infinite product

$$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \tag{8.2.3.2}$$

(called a *Weierstrass product*) converges uniformly on compact subsets of $\mathbb{C}$ to an entire function F. The zeros of F are precisely the points $\{a_n\}$, counted with multiplicity.

8.2.4 Existence of an Entire Function with Prescribed Zeros

Let $\{a_n\}_{n=1}^{\infty}$ be any sequence in the plane with no finite accumulation point. Then there exists an entire function f with zero set precisely equal to $\{a_n\}_{n=1}^{\infty}$ (counting multiplicities). The function f is given by a Weierstrass product.

8.2.5 The Weierstrass Factorization Theorem

Let f be an entire function. Suppose that f vanishes to order m at 0, $m \geq 0$. Let $\{a_n\}$ be the other zeros of f, listed with multiplicities. Then there is an entire function g such that

$$f(z) = z^m \cdot e^{g(z)} \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right). \tag{8.2.5.1}$$

8.3 The Theorems of Weierstrass and Mittag-Leffler

8.3.1 The Concept of Weierstrass's Theorem

Let $U \subset \mathbb{C}$ be a domain. The only necessary condition that we know for a set $\{a_j\} \subseteq U$ to be the zero set of a function f holomorphic on U is that

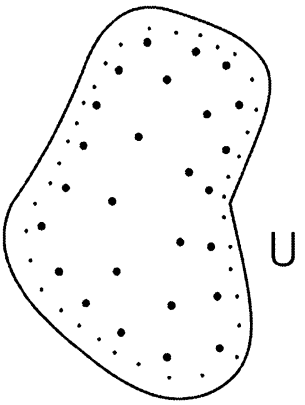

FIGURE 8.1
A discrete set A inside U.

$\{a_j\}$ have no accumulation point in U. It is remarkable that this condition is also sufficient: that is the content of Weierstrass's theorem.

8.3.2 Weierstrass's Theorem

> Let $U \subseteq \mathbb{C}$ be any open set. Let $a_1, a_2, \ldots$ be a finite or infinite sequence in U (possibly with repetitions) that has no accumulation point *in* U. Then there exists a holomorphic function f on U whose zero set is precisely $\{a_j\}$. (The function f is constructed by taking an infinite product.)

We next want to formulate a result about maximal domains of existence (or domains of definition) of holomorphic functions. But first we need a geometric fact about open subsets of the plane.

8.3.3 Construction of a Discrete Set

Let $U \subsetneq \mathbb{C}$ be any open set. Then there exists a countably infinite set $A \subseteq U$ such that

(8.3.3.1) A has no accumulation point in U.

(8.3.3.2) Every $P \in \partial U$ is an accumulation point of A.

See Figure 8.1. Details of this construction appear in [GK, p. 268].

8.3.4 Domains of Existence for Holomorphic Functions

> Let $U \subseteq \mathbb{C}$ be any proper, connected, open subset. There is a function f holomorphic on U such that f cannot be analytically continued past any $P \in \partial U$ (see also §§9.2.2 on the concept of "regular boundary point").

The verification of this last result is just to apply Weierstrass's theorem to the discrete set described in §§8.3.3. This yields a non-constant holomorphic function f on U whose zero set accumulates at every boundary point of U. If f were to analytically continue to any strictly larger open set $\widetilde{U}$, then $\widetilde{U}$ would contain a point of ∂U, hence would have an interior accumulation point of the zeros of f. Thus f would be identically zero, and that would be a contradiction.

8.3.5 The Field Generated by the Ring of Holomorphic Functions

Another important corollary of Weierstrass's theorem is that, for any open U, the field generated by the ring of holomorphic functions on U is the field of all meromorphic functions on U. In simpler language:

> Let $U \subseteq \mathbb{C}$ be open. Let m be meromorphic on U. Then there are holomorphic functions f, g on U such that
>
> $$m(z) = \frac{f(z)}{g(z)}. \tag{8.3.5.1}$$

8.3.6 The Mittag-Leffler Theorem

Since it is possible to prescribe zeros of a holomorphic function on any open U, then of course we can (in principle) prescribe poles—since $1/f$ has poles exactly where f has zeros. But we can do better: with a little extra work we can prescribe the negative power portion of the Laurent series on any discrete subset of U.

We now formulate the basic result on prescribing pole behavior, known as the Mittag-Leffler theorem, in two different (but equivalent) ways: one qualitative and the other quantitative.

Mittag-Leffler Theorem: First Version

> Let $U \subseteq \mathbb{C}$ be any open set. Let $\alpha_1, \alpha_2, \ldots$ be a finite or countably infinite set of *distinct elements of* U with no accumulation

point in U. Suppose, for each j, that U_j is a neighborhood of α_j. Further assume, for each j, that m_j is a meromorphic function defined on U_j with a pole at α_j and no other poles. Then there exists a meromorphic m on U such that $m - m_j$ is holomorphic on U_j for every j.

The Mittag-Leffler Theorem: Alternative Formulation

Let $U \subseteq \mathbb{C}$ be any open set. Let $\alpha_1, \alpha_2, \ldots$ be a finite or countably infinite set of distinct elements of U, having no accumulation point in U. Let s_j be a sequence of Laurent polynomials (or "principal parts"),

$$s_j(z) = \sum_{\ell=-p(j)}^{-1} a_\ell^j \cdot (z - \alpha_j)^\ell \tag{8.3.6.1}$$

(see §§4.3.1). Then there is a meromorphic function on U whose principal part at each α_j is s_j.

8.3.7 Prescribing Principal Parts

The theorems of Weierstrass and Mittag-Leffler can be combined to allow specification of a finite part of the Laurent series at a discrete set of points:

Let $U \subseteq \mathbb{C}$ be an open set and let $\alpha_1, \alpha_2, \ldots$ be a finite or countably infinite set of distinct points of U having no interior accumulation point in U. For each j let there be given an expression

$$t_j(z) = \sum_{\ell=-M(j)}^{N(j)} a_\ell^j \cdot (z - \alpha_j)^\ell, \tag{8.3.7.1}$$

with $M(j), N(j) \geq 0$. Then there is a meromorphic function m on U, holomorphic on $U \setminus \{\alpha_j\}$, such that: if $-M(j) \leq \ell \leq N(j)$, then the ℓ^{th} Laurent coefficient of m at α_j is a_j^ℓ.

8.4 Normal Families

8.4.1 Normal Convergence

A sequence of functions f_j on an open set $U \subseteq \mathbb{C}$ is said to *converge normally* to a limit function f_0 on U (see §§8.1.3) if $\{f_j\}$ converges to f_0

uniformly on compact subsets of U. That is, the convergence is normal if, for each compact set $K \subseteq U$ and each $\epsilon > 0$, there is an $N > 0$ (depending on K and ϵ) such that when $j > N$ and $z \in K$, then $|f_j(z) - f_0(z)| < \epsilon$ (see §§8.1.3, 3.1.5).

The functions $f_j(z) = z^j$ converge normally on the unit disc D to the function $f_0(z) \equiv 0$. The sequence does *not* converge uniformly on all of D to f_0, but does converge uniformly on each compact subset of D.

8.4.2 Normal Families

Let $\mathcal{F}$ be a family of (holomorphic) functions with common domain U. We say that $\mathcal{F}$ is a *normal family* if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U, i.e., *converges normally* on U.

Let $\mathcal{F}$ be a family of functions on an open set $U \subseteq \mathbb{C}$. We say that $\mathcal{F}$ is *bounded* if there is a constant $N > 0$ such that $|f(z)| \leq N$ for all $z \in U$ and all $f \in \mathcal{F}$. We say that $\mathcal{F}$ is *bounded on compact sets* if for each compact set $K \subseteq U$ there is a constant $M = M_K$ such that for all $f \in \mathcal{F}$ and all $z \in K$ we have

$$|f(z)| \leq M. \tag{8.4.2.1}$$

8.4.3 Montel's Theorem, First Version

> Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ be a bounded family of holomorphic functions on an open set $U \subseteq \mathbb{C}$. Then there is a sequence $\{f_j\} \subseteq \mathcal{F}$ such that f_j converges normally on U to a limit (holomorphic) function f_0.

Thus a bounded family of holomorphic functions is normal.

8.4.4 Montel's Theorem, Second Version

> **Montel's Theorem, Second Version** Let $U \subseteq \mathbb{C}$ be an open set and let $\mathcal{F}$ be a family of holomorphic functions on U that is bounded on compact sets. Then there is a sequence $\{f_j\} \subseteq \mathcal{F}$ that converges normally on U to a limit (necessarily holomorphic) function f_0.

Thus a family of holomorphic functions that is bounded on compact sets is normal.

8.4.5 Examples of Normal Families

(8.4.5.1) Consider the family $\mathcal{F} = \{z^j\}_{j=1}^{\infty}$ of holomorphic functions. If we take U to be any subset of the unit disc, then $\mathcal{F}$ is bounded (by

1) so Montel's theorem (first version) guarantees that there is a subsequence that converges uniformly on compact subsets. Of course in this case it is plain by inspection that *any* subsequence will converge uniformly on compact sets to the identically zero function.

The family $\mathcal{F}$ fails to be bounded on compact sets for any U that contains points of modulus greater than one. Thus neither version of Montel's theorem would apply on such a U. And there is no convergent sequence in $\mathcal{F}$ for such a U.

(8.4.5.2) Let $\mathcal{F} = \{z/j\}_{j=1}^{\infty}$ on $\mathbb{C}$. Then there is no bound M such that $|z/j| \leq M$ for all j and all $z \in \mathbb{C}$. But for each fixed compact subset $K \subseteq \mathbb{C}$ there is a constant M_K such that $|z/j| < M_K$ for all j and all $z \in K$. (For instance, $M_K = \max\{|z| : z \in K\}$ would do.) Therefore the second version of Montel's theorem applies. And indeed the sequence $\{z/j\}_{j=1}^{\infty}$ converges normally to 0 on $\mathbb{C}$.

Chapter 9

Applications of Infinite Sums and Products

9.1 Jensen's Formula and an Introduction to Blaschke Products

9.1.1 Blashke Factors

First we need means for manipulating the zeros of a holomorphic function on the disc. What we want is an analogue for the disc of the factors $(z-a)$ that are used when we study polynomials on $\mathbb{C}$. The necessary device is what are called the *Blaschke factors:*

If $a \in D(0,1)$, then we define the Blaschke factor

$$B_a(z) = \frac{z-a}{1-\overline{a}z}. \tag{9.1.1.1}$$

Observe that we have seen these functions before in the guise of Möbius transformations (§§5.5.1, §§6.2.2).

The function B_a is holomorphic on a neighborhood of $\overline{D}(0,1)$. It has a simple zero at a and no others. It has a pole at $1/\overline{a}$. It also satisfies the property that $|B_a(z)| = 1$ if $z \in \partial D$.

The result of §§9.1.2 derives from dividing out the zeros of f using Blaschke factors:

9.1.2 Jensen's Formula

> Let f be holomorphic on a neighborhood of $\overline{D}(0,r)$ and suppose that $f(0) \neq 0$. Let $a_1, \ldots, a_k$ be the zeros of f in $D(0,r)$, counted according to their multiplicities. Assume that f does not vanish on $\partial D(0,r)$. Then

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$$\log|f(0)| + \sum_{j=1}^{k} \log\left|\frac{r}{a_j}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta. \tag{9.1.2.1}$$

9.1.3 Jensen's Inequality

Jensen's Inequality Let f as in §§9.1.2. Observing that $|r/a_j| \geq 1$ hence $\log|r/a_j| \geq 0$, we conclude that

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta. \tag{9.1.3.1}$$

9.1.4 Zeros of a Bounded Holomorphic Function

We now turn to what is a recurring theme in complex function theory: that the growth rate of a holomorphic (or meromorphic) function gives information on the distribution of its zeros. For the unit disc, the result will be a consequence of Jensen's formula.

If f is a non-constant *bounded* holomorphic function on $D(0,1)$ and $a_1, a_2, \ldots$ are the zeros of f (counted according to their multiplicities), then

$$\sum_{j=1}^{\infty}(1 - |a_j|) < \infty. \tag{9.1.4.1}$$

It is remarkable that the converse of the preceding result holds (one might have expected that there would be additional restrictions that need to be imposed on the a_j):

9.1.5 The Blaschke Condition

If $\{a_j\} \subseteq D(0,1)$ (with possible repetitions) satisfies

$$\sum_{j=1}^{\infty}(1 - |a_j|) < \infty \tag{9.1.5.1}$$

and no $a_j = 0$, then there is a bounded holomorphic function on $D(0,1)$ which has zero set consisting precisely of the a_j's, counted according to their multiplicities. Specifically, the infinite product

$$\prod_{j=1}^{\infty} \frac{-\overline{a}_j}{|a_j|} B_{a_j}(z) \tag{9.1.5.2}$$

converges uniformly on compact subsets of $D(0,1)$ to a bounded holomorphic function $B(z)$. The zeros of B are precisely the a_j's, counted according to their multiplicities.

9.1.6 Blaschke Products

Remark: A *Blaschke product* is an expression of the form

$$B(z) = z^m \cdot \prod_{j=1}^{\infty} \frac{-\overline{a_j}}{|a_j|} B_{a_j}(z), \tag{9.1.6.1}$$

where m is a non-negative integer. When $\sum(1 - |a_j|) < \infty$, the result of §§9.1.5 guarantees that the product converges.

9.1.7 Blaschke Factorization

Suppose that f is a bounded holomorphic function on $D(0,1)$ vanishing to order $m \geq 0$ at 0 and that $\{a_j\}$ are its other zeros listed with multiplicities. Then

$$f(z) = z^m \cdot \left\{ \prod_{j=1}^{\infty} -\frac{\overline{a}_j}{|a_j|} B_{a_j}(z) \right\} \cdot F(z) \tag{9.1.7.1}$$

where F is a bounded holomorphic function on $D(0,1)$, F is zero-free, and

$$\sup_{z \in D(0,1)} |f(z)| = \sup_{z \in D(0,1)} |F(z)|. \tag{9.1.7.2}$$

In other words, f is the product of **(i)** a factor of the form z^m, **(ii)** a Blaschke product, and **(iii)** a non-vanishing function.

9.2 The Hadamard Gap Theorem

9.2.1 The Technique of Ostrowski

In this brief section we present a technique of Ostrowski for producing series that exhibit a phenomenon called "over-convergence" (to be defined below).

9.2.2 The Ostrowski-Hadamard Gap Theorem

Let f be holomorphic on a domain U. A point P of ∂U is called *regular* if f extends to be a holomorphic function on an open set containing U and also the point P (Figure 9.1).

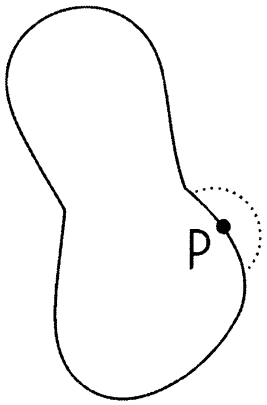

FIGURE 9.1
The point P is regular for f.

Ostrowski-Hadamard Let $0 < p_1 < p_2 < \cdots$ be integers and suppose that there is a $\lambda > 1$ such that

$$\frac{p_{j+1}}{p_j} > \lambda \quad \text{for} \quad j = 1, 2, \ldots. \tag{9.2.2.1}$$

Suppose that, for some sequence of complex numbers $\{a_j\}$, the power series

$$f(z) = \sum_{j=1}^{\infty} a_j z^{p_j} \tag{9.2.2.2}$$

has radius of convergence 1. Then no point of ∂D is regular for f.

The meaning of this theorem is that a function f constructed according to the series expansion (9.2.2.2) will have the disc as its natural domain of definition: such an f cannot be holomorphically extended to any larger open set. See also Chapter 10 on analytic continuation.

9.3 Entire Functions of Finite Order

9.3.1 Rate of Growth and Zero Set

This section contains the version for entire functions of a basic idea that we alluded to earlier: that the rate of growth of a function controls the distribution of its zeros.

9.3.2 Finite Order

An entire function f is said to be of *finite order* if there exist numbers $a, r > 0$ such that

$$|f(z)| \leq \exp(|z|^a) \qquad \text{for all } |z| > r. \tag{9.3.2.1}$$

The infimum of all numbers a for which such an inequality holds is called the *order* of f and is denoted by $\lambda = \lambda(f)$.

Let f be an entire function of finite order λ and satisfying $f(0) = 1$ (this last is a harmless normalization). If $a_1, a_2, \ldots$ are the zeros of f listed with multiplicities and if $|a_1| \leq |a_2| \leq \ldots$, then

$$\sum_{n=1}^{\infty} |a_n|^{-\lambda-1} < \infty. \tag{9.3.2.2}$$

In §§8.2.3 and what followed we showed how to write a factorization of an entire function in a fashion that made its zeros explicit (the "Weierstrass factorization"). Recall that the convergence of the Weierstrass product relied on a condition such as the convergence of the series discussed in §§8.2.3. The discussion below shows how §§8.2.3 links up with (9.3.2.2).

9.3.3 Finite Order and the Exponential Term of Weierstrass

Now that we have examined what $\lambda(f)$ says about the zeros of f we will turn to the other part of the Weierstrass Factorization: the function g. Some technical preliminaries are needed for this purpose.

Let f be a non-constant entire function of finite order λ and suppose that $f(0) = 1$. Let $\{a_1, a_2, \ldots\}$ be the zeros of f listed with multiplicities. Suppose that $|a_1| \leq |a_2| \leq \cdots$. If $p > \lambda - 1$ is an integer, then

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p! \sum_n \frac{1}{(a_n - z)^{p+1}}. \tag{9.3.3.1}$$

9.3.4 Weierstrass Canonical Products

If f is an entire function of finite order λ and $f(0) = 1$, then the canonical Weierstrass product

$$f(z) = e^{g(z)} P(z) \tag{9.3.4.1}$$

has the property that g is a polynomial of degree less than or equal to λ.

9.3.5 The Hadamard Factorization Theorem

If f is an entire function of finite order and $\{a_j\}$ the zeros of f, listed with multiplicity, then we define the *rank* p of f to be the least positive

integer such that

$$\sum_{a_n \neq 0} |a_n|^{-(p+1)} < \infty. \tag{9.3.5.1}$$

Observe that the result of §§9.3.2 guarantees that such a p exists. Now Weierstrass's theorem (§§9.3.4) tells us that the function g in the exponential is a polynomial of degree q with $q \leq \lambda$. We define the *genus* μ of f to be the maximum of p and q.

An entire function of finite order λ is also of finite genus μ and

$$\mu \leq \lambda. \tag{9.3.5.2}$$

9.3.6 Value Distribution Theory

Let f be an entire function of finite order $\lambda \notin \mathbb{Z}$. If c lies in the image of f, then there are infinitely many distinct $z_j \in \mathbb{C}$ such that $f(z_j) = c$.

Let f be a non-constant entire function of finite order. Then the image of f contains all complex numbers except possibly one (this is Picard's little theorem, §§10.5.2). If, in addition, f has non-integral order, then f assumes each of these values infinitely many times.

Chapter 10

Analytic Continuation

10.1 Definition of an Analytic Function Element

10.1.1 Continuation of Holomorphic Functions

Suppose that V is a connected, open subset of $\mathbb{C}$ and that $f_1 : V \to \mathbb{C}$ and $f_2 : V \to \mathbb{C}$ are holomorphic functions. If there is an open, non-empty subset U of V such that $f_1 \equiv f_2$ on U, then $f_1 \equiv f_2$ on all of V (see §§3.2.3). Put another way, if we are given an f holomorphic on U, then there is at most one way to extend f to V so that the extended function is holomorphic. [Of course there might not even be one such extension: if V is the unit disc and U the disc $D(3/4, 1/4)$, then the function $f(z) = 1/z$ does not extend. Or if U is the plane with the non-positive real axis removed, $V = \mathbb{C}$, and $f(re^{i\theta}) = r^{1/2}e^{i\theta/2}$, $-\pi < \theta < \pi$, then again no extension from U to V is possible.]

This chapter deals with the question of when this extension process can be carried out, and in particular what precise meaning can be given to extending f from U to as large a set V as possible.

10.1.2 Examples of Analytic Continuation

We introduce the basic issues of "analytic continuation" by way of three examples:

EXAMPLE 10.1.2.1 Define

$$f(z) = \sum_{j=0}^{\infty} z^j. \tag{10.1.2.1.1}$$

This series converges normally on the disc $D = \{z \in \mathbb{C} : |z| < 1\}$. It diverges for $|z| > 1$. Is it safe to say that D is the natural domain of definition for

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f (refer to §§8.3.2, §§8.3.4 for this terminology)? Or can we "continue" f to a larger open set?

We cannot discern easily the answer to this question simply by examining the power series. Instead, we should sum the series and observe that

$$f(z) = \frac{1}{1-z}. \tag{10.1.2.1.2}$$

This formula for f agrees with the original definition of f as a series; however, the formula makes sense for all $z \in \mathbb{C}\backslash\{1\}$. In our new terminology, to be made more precise later, f has an analytic continuation to $\mathbb{C} \setminus \{1\}$.

Thus we see that the natural domain of definition for f is the rather large set $\mathbb{C} \setminus \{1\}$. However, the original definition, by way of a series, gave little hint of this fact. □

EXAMPLE 10.1.2.2 Consider the function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt. \tag{10.1.2.2.1}$$

This function is known as the gamma function of Euler; it is discussed in detail in §§13.1.1. Let us make the following quick observations:

(10.1.2.2.2) The term t^{z-1} has size $|t^{z-1}| = t^{\operatorname{Re} z-1}$. Thus the singularity at the origin will be integrable when $\operatorname{Re} z > 0$.

(10.1.2.2.3) Because of the presence of the exponential factor, the integrand will certainly be integrable at infinity.

(10.1.2.2.4) The function Γ is holomorphic on the domain $U_0 \equiv \{z : \operatorname{Re} z > 0\}$: the functions

$$\int_a^{1/a} t^{z-1}e^{-t}\,dt, \tag{10.1.2.2.5}$$

with $a > 0$, are holomorphic by differentiation under the integral sign (or use Morera's theorem—§§2.3.2), and $\Gamma(z)$ is the normal limit of these integrals as $a \to 0^+$.

The given definition (10.1.2.2.1) of $\Gamma(z)$ makes no sense when $\operatorname{Re} z \leq 0$ because the improper integral diverges at 0. Can we conclude from this observation that the natural domain of definition of Γ is U_0?

Let us examine this question by integrating by parts:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt = \frac{1}{z}t^z e^{-t}\Big|_0^\infty + \frac{1}{z}\int_0^\infty t^z e^{-t}\,dt. \tag{10.1.2.2.6}$$

An elementary analysis shows that, as long as $\operatorname{Re} z \neq 0$, the boundary terms vanish (in the limit). Thus we see that

$$\Gamma(z) = \frac{1}{z}\int_0^\infty t^z e^{-t}\,dt. \tag{10.1.2.2.7}$$

Now, whereas the original definition (10.1.2.2.1) of the gamma function made sense on U_0, this new formula (which agrees with the old one on U_0) actually makes sense on $U_1 \equiv \{z : \operatorname{Re} z > -1\} \setminus \{0\}$. No difficulty about the convergence of the integral as the lower limit tends to 0^+ occurs if $z \in \{z : \operatorname{Re} z > -1\}$.

We can integrate by parts once again, and find that

$$\Gamma(z) = \frac{1}{z(z+1)}\int_0^\infty t^{z+1} e^{-t}\,dt. \tag{10.1.2.2.8}$$

This last formula makes sense on $U_2 = \{z : \operatorname{Re} z > -2\} \setminus \{0, -1\}$.

Continuing this process, we may verify that the gamma function, originally defined only on U_0, can be "analytically continued" to $U = \{z \in \mathbb{C} : z \neq 0, -1, -2, \ldots\}$. □

In the first two examples, the functions are given by a formula that only makes sense on a certain open set; yet there is in each case a device for extending the function to a larger open set. Recall that, by our uniqueness results for analytic functions, there can be at most one way to effect this "analytic continuation" process to a fixed, larger (connected) open set. In the next example, we learn about possible ambiguities in the process when one attempts continuation along two different paths.

EXAMPLE 10.1.2.3 Consider the function $f(z)$, initially defined on the disc $D((1 + i0), 1/2)$ by $f(re^{i\theta}) = r^{1/2}e^{i\theta/2}$. Here it is understood that $-\pi/4 < \theta < \pi/4$. This function is well-defined and holomorphic; in fact it is the function usually called the *principal branch* of $\sqrt{z}$. Note that $[f(z)]^2 = z$.

Imagine analytically continuing f to a second disc, as shown in Figure 10.1. This is easily done, using the same definition $f(re^{i\theta}) = r^{1/2}e^{i\theta/2}$. If we continue to a third disc (Figure 10.2), and so on, we end up defining the square root function at $z = -1$. See Figure 10.3. Indeed, we find that $f(-1)$ has the value i.

However, we might have begun our analytic continuation process as shown in Figure 10.4. We begin at 1, and iterate the continuation process in a clockwise direction so that process ends at $z = -1$. Doing so, we would have found that $f(-1) = -i$.

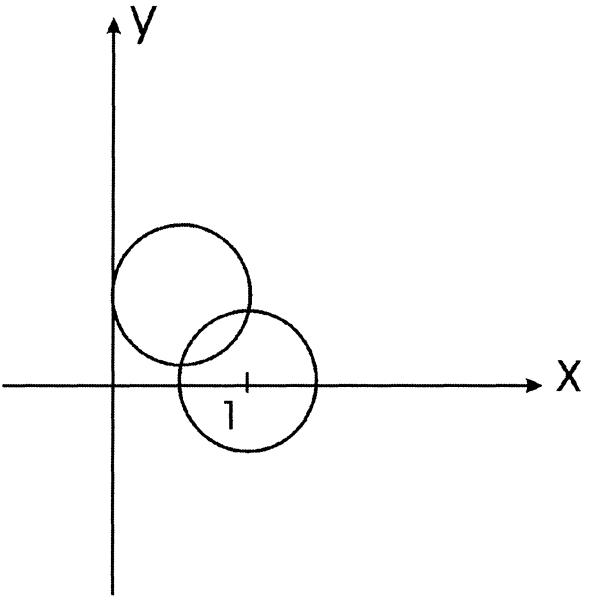

FIGURE 10.1
Analytic continuation to a second disc.

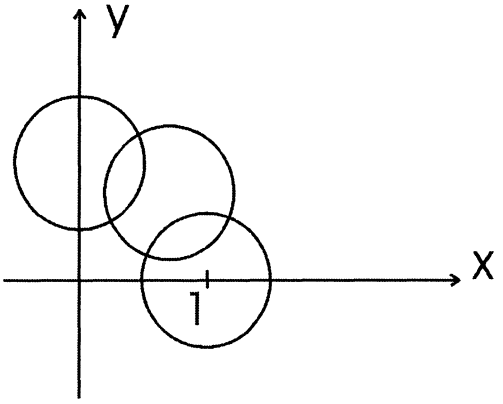

FIGURE 10.2
Analytic continuation to a third disc.

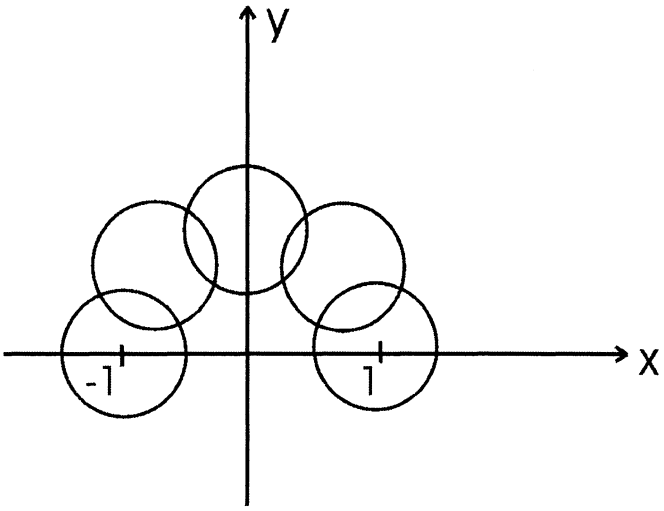

FIGURE 10.3
Defining the square root function at -1.

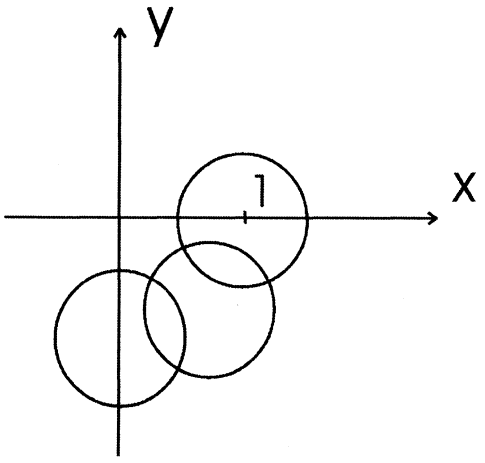

FIGURE 10.4
Analytic continuation in the other direction.

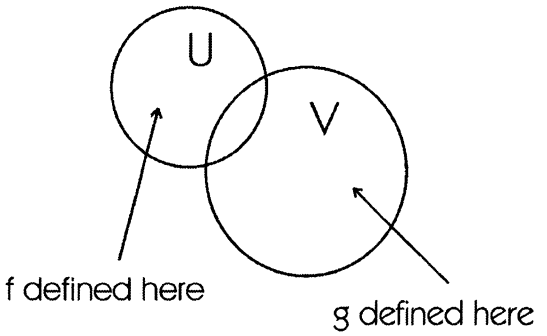

FIGURE 10.5
The function element (g, V) is a direct analytic continuation of (f, U).

Thus we see that the process of analytic continuation can be ambiguous. In the present example, the ambiguity is connected to the fact that a holomorphic square root function cannot be defined in any neighborhood of the origin; yet our two paths of analytic continuation encircle the origin. □

It is because of the phenomena illustrated in these three examples—and many others like them—that we must take a detailed and technical approach to the process of analytic continuation. Even making the questions themselves precise takes some thought.

10.1.3 Function Elements

A *function element* is an ordered pair (f, U), where U is a disc $D(P, r)$ and f is a holomorphic function defined on U. If W is an open set, then a function element in W is a pair (f, U) such that $U \subseteq W$.

10.1.4 Direct Analytic Continuation

Let (f, U) and (g, V) be function elements. We say that (g, V) is a *direct analytic continuation* of (f, U) if $U \cap V \neq \emptyset$ and f and g are equal on $U \cap V$ (Figure 10.5). Obviously (g, V) is a direct analytic continuation of (f, U) if and only if (f, U) is a direct analytic continuation of (g, V).

10.1.5 Analytic Continuation of a Function

If $(f_1, U_1), \ldots, (f_k, U_k)$ are function elements and if each (f_j, U_j) is a direct analytic continuation of (f_{j-1}, U_{j-1}), $j = 2, \ldots, k$, then we say that (f_k, U_k) is an *analytic continuation* of (f_1, U_1) (Figure 10.6).

Clearly (f_k, U_k) is an analytic continuation of (f_1, U_1) if and only if

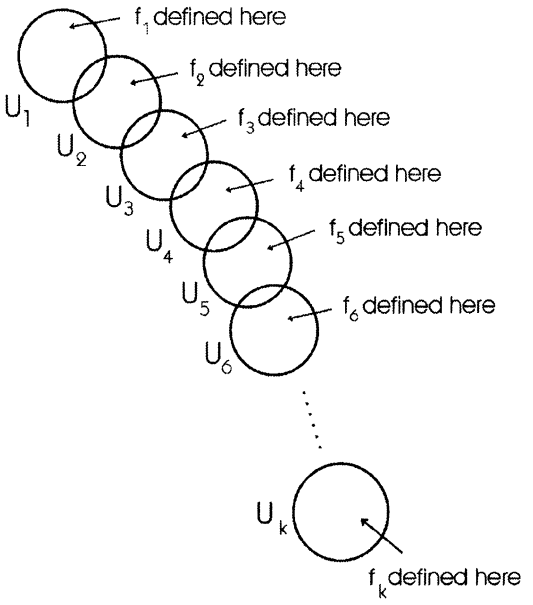

FIGURE 10.6
The function element (f_k, U_k) *is an analytic continuation of* (f_1, U_1).

(f_1, U_1) is an analytic continuation of (f_k, U_k). Also if (f_k, U_k) is an analytic continuation of (f_1, U_1) via a chain $(f_1, U_1), \ldots, (f_k, U_k)$ and if $(f_{k+\ell}, U_{k+\ell})$ is an analytic continuation of (f_k, U_k) via a chain $(f_k, U_k), (f_{k+1}, U_{k+1}), \ldots$ $(f_{k+\ell}, U_{k+\ell})$, then stringing the two chains together into $(f_1, U_1), \ldots, (f_{k+\ell}, U_{k+\ell})$ exhibits $(f_{k+\ell}, U_{k+\ell})$ as an analytic continuation of (f_1, U_1). Obviously any function element (f, U) is an analytic continuation of itself.

10.1.6 Global Analytic Functions

Thus we have an equivalence relation (see [KRA3, p. 52] or the Glossary for this terminology) by way of analytic continuation on the set of function elements: namely, two function elements are equivalent if one is the analytic continuation of the other. The equivalence classes ([KRA3, p. 53]) induced by this relation are called *(global) analytic functions.* However, a caution is in order: global analytic functions are not yet functions in the usual sense, and they are not analytic in any sense that we have defined as yet. Justification for the terminology will appear in due course.

Note that the initial element $(f, U) = (f_1, U_1)$ uniquely determines the global analytic function, or equivalence class, that contains it. But a global analytic function may include more than one function element of the form (f, U) *for a fixed disc* U (see Example 10.1.2.3). Indeed, a global analytic

function f may have in effect more than one value at a point: two function elements (f_1, U) and (f_2, U) can be equivalent even though $f_1(P) \neq f_2(P)$, where P is the center of the disc U. If $\mathbf{f}$ denotes the global analytic function corresponding to (f, U), then we call (f, U) a *branch* of $\mathbf{f}$.

10.1.7 An Example of Analytic Continuation

EXAMPLE 10.1.7.1 Let $U = D(1 + i0, 1/2)$ and let f be the holomorphic function $\log z$. Here $\log z$ is understood to be defined to be $\log|z| + i\arg z$, and $-\pi/4 < \arg z < \pi/4$. As in Example 10.1.2.3, the function element (f, U) can be analytically continued to the point $-1 + i0$ in (at least) two different ways, depending on whether the continuation is along a curve proceeding clockwise about the origin or counterclockwise about the origin.

In fact, all the "branches" of $\log z$ can be obtained by analytic continuation of the $\log|z| + i\arg z$ branch on $D(1 + 0i, 1/2)$ (by continuing several times around, either clockwise or counterclockwise). Thus the idea of "branches" of $\log z$ (§§4.5.6, §§10.1.2) is consistent with the general analytic continuation terminology just introduced. □

In some situations, it is convenient to think of a function element as a convergent power series. Then the role of the open disc U is played by the domain of convergence of the power series. This is a useful heuristic idea for the reader to bear in mind. From this viewpoint, two function elements (f_1, U) and (f_2, V) at a point P (such that U and V are discs centered at the same point P) should be regarded as equal if $f_1 \equiv f_2$ on $U \cap V$.

10.2 Analytic Continuation along a Curve

10.2.1 Continuation on a Curve

Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve and let (f, U) be a function element with $\gamma(0)$ the center of the disc U (Figure 10.7). An *analytic continuation* of (f, U) along the curve γ is a collection of function elements (f_t, U_t), $t \in [0, 1]$, such that

(10.2.1.1) $(f_0, U_0) = (f, U)$.

(10.2.1.2) For each $t \in [0, 1]$, the center of the disc U_t is $\gamma(t)$, $0 \leq t \leq 1$.

(10.2.1.3) For each $t \in [0, 1]$, there is an $\epsilon > 0$ such that, for each $t' \in [0, 1]$ with $|t' - t| < \epsilon$, it holds that

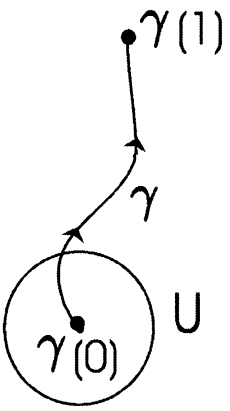

FIGURE 10.7
Analytic continuation along a curve.

(a) $\gamma(t') \in U_t$ and hence $U_{t'} \cap U_t \neq \emptyset$;

(b) $f_t \equiv f_{t'}$ on $U_{t'} \cap U_t$ (so that (f_t, U_t) is a direct analytic continuation of $(f_{t'}, U_{t'})$). Refer to Figure 10.8.

Let (f, U) be a function element with U a disc having center P. Let γ be a curve such that $\gamma(0) = P$. Any two analytic continuations of (f, U) along γ agree in the following sense: if (f_1, U_1) is the terminal element of one analytic continuation (f_t, U_t) and if $(\widetilde{f}_1, \widetilde{U}_1)$ is the terminal element of another analytic continuation $(\widetilde{f}_t, \widetilde{U}_t)$, then f_1 and $\widetilde{f}_1$ are equal on $U_1 \cap \widetilde{U}_1$.

10.2.2 Uniqueness of Continuation along a Curve

Thus we see that the analytic continuation of a given function element along a given curve is essentially unique, if it exists. From here on, to avoid being pedantic, we shall regard two analytic continuations (f_t, U_t) and $(\widetilde{f}_t, \widetilde{U}_t)$ as "equal," or equivalent, if $f_t = \widetilde{f}_t$ on $U_t \cap \widetilde{U}_t$ for all t. With this terminological convention (which will cause no trouble), the result of §10.2.1 says exactly that analytic continuation of a given function element along a curve is unique.

10.3 The Monodromy Theorem

The fundamental issue to be addressed in the present section is this:

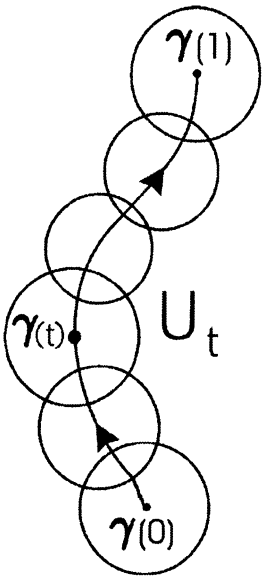

FIGURE 10.8
Details of analytic continuation along a curve.

10.3.1 Unambiguity of Analytic Continuation

> Let P and Q be points in the complex plane. Let (f, U) be a function element such that U is a disc centered at P. If γ_1, γ_2 are two curves that begin at P and terminate at Q then does the terminal element of the analytic continuation of (f, U) along γ_1 equal the terminal element of the analytic continuation of (f, U) along γ_2 (on their common domain of definition)?

We shall begin to answer this question in §§10.3.2. The culmination of our discussion will be the monodromy theorem in §§10.3.5.

10.3.2 The Concept of Homotopy

Let W be a domain in $\mathbb{C}$. Let $\gamma_0 : [0,1] \to W$ and $\gamma_1 : [0,1] \to W$ be curves. Assume that $\gamma_0(0) = \gamma_1(0) = P$ and that $\gamma_0(1) = \gamma_1(1) = Q$. We say that γ_0 and γ_1 are *homotopic in* W (with fixed endpoints) if there is a continuous function

$$H : [0,1] \times [0,1] \to W \tag{10.3.2.1}$$

such that

(10.3.2.2) $H(0,t) = \gamma_0(t)$ for all $t \in [0,1]$;

(10.3.2.3) $H(1,t) = \gamma_1(t)$ for all $t \in [0,1]$;

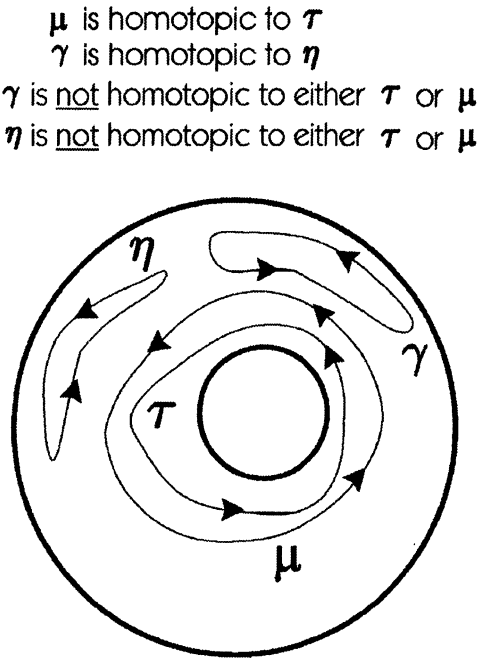

FIGURE 10.9
The concept of homotopy.

(10.3.2.4) $H(s,0) = P$ for all $s \in [0,1]$;

(10.3.2.5) $H(s,1) = Q$ for all $s \in [0,1]$.

Then H is called a *homotopy* (with fixed endpoints) of the curve γ_0 to the curve γ_1. Refer to Figure 10.9 to view two curves that are homotopic and two that are not.

10.3.3 Fixed Endpoint Homotopy

Note: Since we are only interested in homotopies with fixed endpoints, we shall omit the phrase "with fixed endpoints" in the remainder of our discussion.

Intuitively, we think of a homotopy H as follows. Let $H_s(t) = H(s,t)$. Then condition (10.3.2.2) says that H_0 is the curve γ_0. Condition (10.3.2.3) says that H_1 is the curve γ_1. Condition (10.3.2.4) says that all the curves H_s begin at P. Condition (10.3.2.5) says that all the curves H_s terminate at Q. The homotopy amounts to a continuous deformation of γ_0 to γ_1 with *all curves in the process restricted to lie in* W.

We introduce one last piece of terminology:

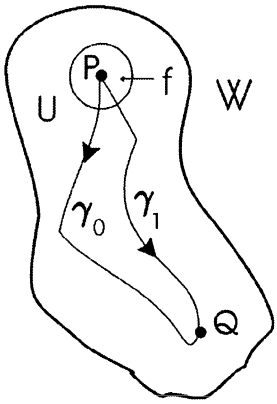

FIGURE 10.10
The monodromy theorem.

10.3.4 Unrestricted Continuation

Let W be a domain and let (f, U) be a function element in W. We say (f, U) admits *unrestricted continuation* in W if there is an analytic continuation (f_t, U_t) of (f, U) along every curve γ that begins at P and lies in W.

10.3.5 The Monodromy Theorem

One situation, in practice the primary situation, in which the question raised in §§10.3.1 always has an affirmative answer is given by the following theorem:

> **The Monodromy Theorem:** Let $W \subseteq \mathbb{C}$ be a connected open set. Let (f, U) be a function element with $U \subseteq W$. Let P denote the center of the disc U. Assume that (f, U) admits unrestricted continuation in W. If γ_0, γ_1 are each curves that begin at P, terminate at some point Q, and are homotopic in W, then the analytic continuation of (f, U) to Q along γ_0 equals the analytic continuation of (f, U) to Q along γ_1. Refer to Figure 10.10.

10.3.6 Monodromy and Globally Defined Analytic Functions

Let $W \subseteq \mathbb{C}$ be a connected open set. Assume further that W is *topologically simply connected*, in the sense that any two curves that begin at the same point and end at the same point (possibly different from the initial

point) are homotopic—see the related discussion of simple connectivity in §§2.3.3. Assume that (f, U) admits unrestricted continuation in W. Then there is a globally defined holomorphic function F on W that equals f on U.

In view of the monodromy theorem, we now can understand specifically how it can be that the function $\sqrt{z}$, and more generally the function $\log z$, cannot be analytically continued in a well-defined fashion to all of $\mathbb{C} \setminus \{0\}$. The difficulty is with the two curves specified in Example 10.1.2.3, or in Example of 10.1.7.1: they are not homotopic *in the region* $\mathbb{C} \setminus \{0\}$ (which is the region of definition of the analytic function being considered).

10.4 The Idea of a Riemann Surface

10.4.1 What is a Riemann Surface?

In this section we give an intuitive description of the concept of what is called a Riemann surface.

The idea of a Riemann surface is that one can visualize geometrically the behavior of function elements and their analytic continuations. At the moment, a global analytic function is an analytic object. A global analytic function is the set of all function elements obtained by analytic continuation along curves (from a base point $P \in \mathbb{C}$) of a function element (f, U) at P. Such a set, which amounts to a collection of convergent power series at different points of the plane $\mathbb{C}$, does not seem very geometric in any sense. But in fact it can be given the structure of a surface, in the intuitive sense of that word, quite easily. (The precise definition and detailed definition of what a "surface" is would take us too far afield: we shall be content here with the informal idea that a surface is a two-dimensional object that locally "looks like" an open set in the plane. A more precise definition would be that a surface is a topological space that is locally homeomorphic to $\mathbb{C}$. See [LST], [ONE] for a more detailed discussion of surfaces.)

10.4.2 Examples of Riemann Surfaces

The idea that we need is most easily appreciated by first working with a few examples. Consider the function element (f, U) defined on $U = D(1 + 0i, 1/2)$ by

$$z = re^{i\theta} \mapsto r^{1/2}e^{i\theta/2}, \tag{10.4.2.1}$$

where $r > 0$ and $-\pi/4 < \theta < \pi/4$ makes the $re^{i\theta}$ representation of $z \in D(1, 1)$ unique. This function element is the "principal branch of $\sqrt{z}$ " at $z = 1$ that we have already discussed in Example 10.1.2.3. The functional

element (f, U) can be analytically continued along every curve γ emanating from 1 and lying in $\mathbb{C}\setminus\{0\}$. Let us denote by $\mathcal{R}$ (for "Riemann surface") the totality of all function elements obtained by such analytic continuations. Of course, in set-theoretic terms, $\mathcal{R}$ is just the global analytic function $\sqrt{z}$, just as we defined this concept earlier in §§10.1.6. All we are trying to do now is to "visualize" $\mathcal{R}$ in some sense.

Note that every point of $\mathcal{R}$ "lies over" a unique point of $\mathbb{C} \setminus \{0\}$. A function element $(f, U) \in \mathcal{R}$ is associated to the center of U, that is to say, (f, U) is a function element at a point of $\mathbb{C} \setminus \{0\}$. So we can define a "projection" $\pi : \mathcal{R} \to \mathbb{C} \setminus \{0\}$ by

$$\pi((f, U)) = \text{the center of the disc } U. \tag{10.4.2.2}$$

This is just new terminology for a situation that we have already discussed.

The projection π of $\mathcal{R}$ is two-to-one onto $\mathbb{C} \setminus \{0\}$. In a neighborhood of a given $z \in \mathbb{C} \setminus \{0\}$, there are exactly two holomorphic branches of $\sqrt{z}$. [If one of these is (f, U), then the other is $(-f, U)$. But there is no way to decide which of (f, U) and $(-f, U)$ is *the* square root in any sense that can be made to vary continuously over all of $\mathbb{C} \setminus \{0\}$.] We can think of $\mathcal{R}$ as a "surface" in the following manner:

Let us define neighborhoods of "points" (f, U) in $\mathcal{R}$ by declaring a neighborhood of (f, U) to be

$$\{(f_p, U_p) : p \in U \text{ and } (f_p, U_p) \text{ is a direct analytic continuation of } (f, U) \text{ to } p\}. \tag{10.4.2.3}$$

This new definition may seem formalistic and awkward. But it has the attractive property that it makes $\pi : \mathcal{R} \to \mathbb{C} \setminus \{0\}$ locally one-to-one. Every (f, U) has a neighborhood that maps, under π, one-to-one onto an open subset of $\mathbb{C} \setminus \{0\}$. This gives a way to think of $\mathcal{R}$ as being locally like an open set in the plane.

Let us try to visualize $\mathcal{R}$ still further. Let $W = \mathbb{C} \setminus \{z = x + i0 : x \leq 0\}$. Then $\pi^{-1}(W)$ decomposes naturally into two components, each of which is an open set in $\mathcal{R}$. (Since we have defined neighborhoods of points in $\mathcal{R}$, we naturally have a concept of open set in $\mathcal{R}$ as well.) These two components are "glued together" in $\mathcal{R}$ itself: $\mathcal{R}$ is connected, while $\pi^{-1}(W)$ is not. Note that, on each of the connected components of $\pi^{-1}(W)$, the projection π is one-to-one. All of this language is just a formalization of the fact that, on W, there are two holomorphic branches of $\sqrt{z}$—namely $re^{i\theta} \mapsto r^{1/2}e^{i\theta/2}$ and $re^{i\theta} \mapsto -r^{1/2}e^{i\theta/2}$, $-\pi < \theta < \pi$.

Each of the components of $\pi^{-1}(W)$ can be thought of as a "copy" of W, since π maps a given component one-to-one onto W. See Figure 10.11. How are these "copies," say $\mathcal{Q}_1$ and $\mathcal{Q}_2$, glued together to form $\mathcal{R}$? We join the second quadrant edge of $\mathcal{Q}_1$ to the third quadrant edge of $\mathcal{Q}_2$ and the second quadrant edge of $\mathcal{Q}_2$ to the third quadrant edge of $\mathcal{Q}_1$. Of course

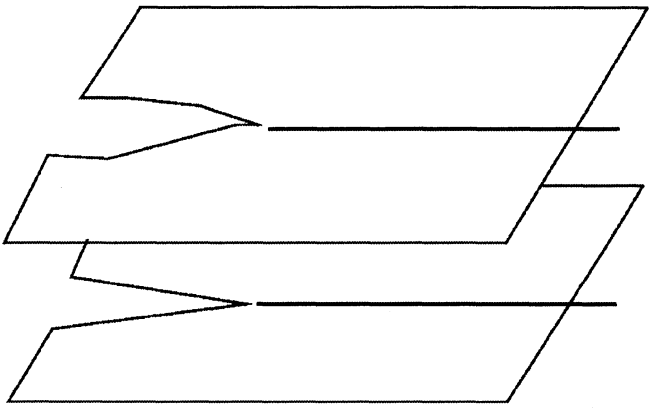

FIGURE 10.11
The Riemann surface for the square root of z.

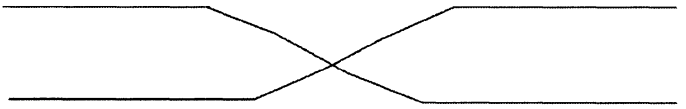

FIGURE 10.12
The joins in the Riemann surface for the square root of z.

these joins cannot be simultaneously performed in three-dimensional space. So our picture is idealized. See Figure 10.12. Tacitly, in our construction of $\mathcal{R}$, we have restored the negative real axis.

10.4.3 The Riemann Surface for the Square Root Function

We have now constructed a surface, known as the "Riemann surface for the function $\sqrt{z}$." This surface that we have obtained by gluing together the two copies of W is in fact homeomorphic to the topological space that we made from $\mathcal{R}$ (the set of function elements) when we defined neighborhoods in $\mathcal{R}$. So we can regard our geometric surface, built from gluing the two copies of W together, and the function element space $\mathcal{R}$, as being the same thing, that is the same surface.

10.4.4 Holomorphic Functions on a Riemann Surface

Since $\pi : \mathcal{R} \to \mathbb{C} \setminus \{0\}$ is locally one-to-one, we can even use this projection to describe what it means for a function $F : \mathcal{R} \to \mathbb{C}$ to be holomorphic. Namely, F is holomorphic if $F \circ \pi^{-1} : \pi(U) \to \mathbb{C}$ is holomorphic for each open set U in $\mathcal{R}$ with π one-to-one on U. With this definition in mind, $f(z) = \sqrt{z}$ becomes a well-defined, "single-valued" holomorphic function

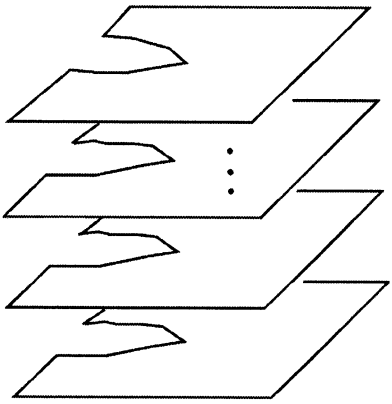

FIGURE 10.13
The Riemann surface for the n^{th} root of z.

on $\mathcal{R}$. To wit, if (f, U) is a function element in $\mathcal{R}$, located at $P \in \mathbb{C} \setminus \{0\}$, i.e., with $\pi(f, U) = P$, then we set

$$F((f, U)) = f(P). \tag{10.4.4.1}$$

In this setup, $F^2((f,U)) = \pi(f^2, U) = P$ [since $f^2(z) = z$]. Therefore F is the square root function, in the sense described.

There are similar pictures for $\sqrt[n]{z}$—see Figure 10.13. Note that the Riemann surface for $\sqrt[n]{z}$ has n sheets, joined together in sequence.

10.4.5 The Riemann Surface for the Logarithm

It requires some time, and some practice, to become familiar with the construction of a Riemann surface from a given function. To get accustomed to it further, let us now discuss briefly the Riemann surface for "$\log z$" (see also Example 10.1.7.1). More precisely, we begin with the "principal branch" $re^{i\theta} \mapsto \log r + i\theta$ defined on $D(1 + 0i, 1/2)$ by requiring that $-\pi/4 < \theta < \pi/4$ and we consider all its analytic continuations along curves emanating from 1. We can visualize the "branches" here by noting that, again with $W = \mathbb{C} \setminus (\{0\} \cup$ the negative real axis), $\pi^{-1}(W)$ has infinitely many components—each a copy of W—on which π maps one-to-one onto W. Namely, these components are the "branches" of $\log z$ on W:

$$re^{i\theta} \mapsto \log r + i\theta + 2\pi ik, \qquad k \in \mathbb{Z}, \tag{10.4.5.1}$$

where $-\pi < \theta < \pi$. Picture each of these (infinitely many) images stacked one above the other (Figure 10.14). We join them in an infinite spiral, or screw, with the upper edge of the k^{th} surface being joined to the lower edge of the $(k-1)^{\text{st}}$ surface. Observe that going around the origin (counted

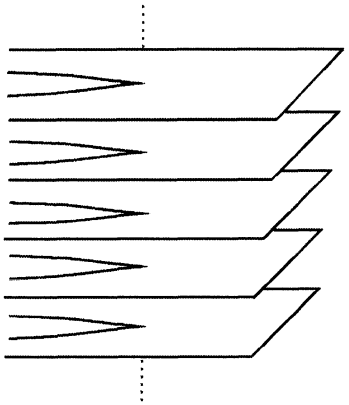

FIGURE 10.14
The Riemann surface for $\log z$.

clockwise in $\mathbb{C} \setminus \{0\}$) corresponds to going around and up one level on the spiral surface. This is the geometric representation of the fact that, when we analytically continue a branch of $\log r + i\theta + 2\pi ik$ once around the origin counterclockwise then k increases by 1. This time there is no joining of the first and last "sheets." The spiral goes on without limit in both direction.

10.4.6 Riemann Surfaces in General

The idea that we have been discussing, of building surfaces from function elements, can be carried out in complete generality: Consider the set of all analytic function elements that can be obtained by analytic continuation (along some curve in $\mathbb{C}$) of a given function element (f, U). This is what we called earlier a global analytic function. Then this set of function elements can actually (and always) be regarded as a connected surface: There is a projection onto an open set in $\mathbb{C}$ obtained by sending each function element to the point of $\mathbb{C}$ at which it is located. This projection is a local identification of the set of function elements with part of $\mathbb{C}$, and so it in effect exhibits the set of function elements as being two-dimensional, i.e., a surface; after this observation, everything proceeds as in the examples. The reader is invited to experiment with these new ideas—see particularly the discussion and exercises in [GK].

10.5 Picard's Theorems

10.5.1 Value Distribution for Entire Functions

The image, or set of values, of an entire function must be quite large. This statement is true in a variety of technical senses, and Nevanlinna theory gives a detailed development of the concept. Here we simply enunciate the theorems of Picard which give some sense of the robustness of entire functions.

10.5.2 Picard's Little Theorem

> **Theorem:** If the range of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ omits two points of $\mathbb{C}$, then f is constant.

The entire function $f(z) = e^z$ shows that an entire function *can* omit one value (in this case, the value 0). But the theorem says that the only way that it can omit two values is if the function in question is constant.

10.5.3 Picard's Great Theorem

The following theorem, known as the great theorem of Picard, strengthens the Little Theorem (see §§10.5.4 for the explication of this connection).

> **Theorem:** Let U be a region in the plane, $P \in U$, and suppose that f is holomorphic on $U \setminus \{P\}$ and has an essential singularity at P. If $\epsilon > 0$, then the restriction of f to $U \cap [D(P, \epsilon) \setminus \{P\}]$ assumes all complex values except possibly one.

10.5.4 The Little Theorem, the Great Theorem, and the Casorati-Weierstrass Theorem

Compare Picard's great theorem with the Casorati/Weierstrass theorem (§§4.1.6). The Casorati/Weierstrass theorem says that, in a deleted neighborhood of an essential singularity, a holomorphic function assumes a dense set of values. Picard's theorem refines this to "all values except possibly one."

What is the connection between the great theorem and the little theorem? And how do these two theorems relate to the results about entire functions that we have already seen?

A non-constant entire function cannot be bounded near infinity, or else it would be bounded on $\mathbb{C}$ and hence (by Liouville's theorem—§§3.1.3) be constant. So it has either a pole or an essential singularity at infinity. In

the first instance, the function is a polynomial (see §§4.6.6). But then the Fundamental Theorem of Algebra (§§1.1.7) tells us that the function assumes *all* complex values. In the second instance, the great theorem applies at the point ∞ and implies the little theorem.

Nevanlinna theory is an analytic refinement of the ideas we have been discussing here. There is a delicate interplay between the rate of growth (at ∞) of an entire function and the distribution of its zeros. Section 9.3 explores some of these connections.

Chapter 11

Rational Approximation Theory

11.1 Runge's Theorem

11.1.1 Approximation by Rational Functions

A *rational* function is, by definition, a quotient of polynomials.

Let $K \subseteq \mathbb{C}$ be compact and let $f : K \to \mathbb{C}$ be a given function on K. Under what conditions is f the uniform limit of rational functions with poles in $\widehat{\mathbb{C}} \setminus K$ (see §§6.3.1-§§6.3.3 for this notation)? There are some obvious necessary conditions:

(11.1.1.1) f must be continuous on K;

(11.1.1.2) f must be holomorphic on $\overset{\circ}{K}$, the interior of K.

It is a striking result of Mergelyan that, in case $\widehat{\mathbb{C}} \setminus K$ has finitely many connected components, then these conditions are also sufficient. In the present section we shall begin by discussing a weaker result—known as Runge's theorem—that is considerably easier to establish.

11.1.2 Runge's Theorem

> Let $K \subseteq \mathbb{C}$ be compact. Let f be holomorphic on a *neighborhood* of K. Let $\mathcal{P} \subseteq \widehat{\mathbb{C}} \setminus K$ contain at least one point from each connected component of $\widehat{\mathbb{C}} \setminus K$. Then for any $\epsilon > 0$ there is a rational function $r(z)$ with poles in $\mathcal{P}$ such that
>
> $$\max_{z \in K} |f(z) - r(z)| < \epsilon. \tag{11.1.2.1}$$

S. G. Krantz, *Handbook of Complex Variables*
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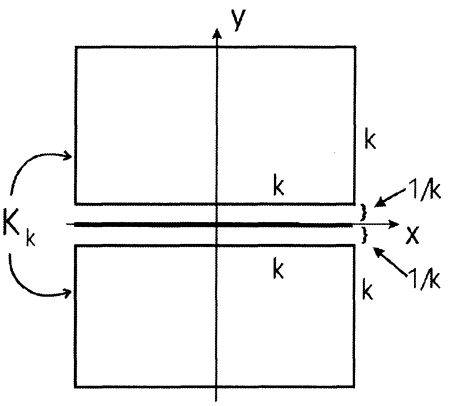

FIGURE 11.1
The region K_k in Example 11.1.4.1.

11.1.3 Approximation by Polynomials

Let $K \subseteq \mathbb{C}$ be compact and assume that $\widehat{\mathbb{C}} \setminus K$ is connected. Let f be holomorphic on a neighborhood of K. Then for any $\epsilon > 0$ there is a holomorphic polynomial $p(z)$ such that

$$\max_K |p(z) - f(z)| < \epsilon. \tag{11.1.3.1}$$

This result follows from Runge's theorem by taking $\mathcal{P} = \{\infty\}$.

11.1.4 Applications of Runge's Theorem

EXAMPLE 11.1.4.1 There is a sequence of holomorphic polynomials p_j such that $p_j \to 1$ normally in the upper half plane, $p_j \to -1$ normally in the lower half plane, and $p_j \to 0$ normally on the real axis. To see this, let

$$K_k = \left\{ z \in \mathbb{C} : \frac{1}{k} \leq |\operatorname{Im} z| \leq k, |\operatorname{Re} z| \leq k \right\} \cup \{ z \in \mathbb{C} : \operatorname{Im} z = 0, |\operatorname{Re} z| \leq k \}. \tag{11.1.4.1.1}$$

See Figure 11.1.

Let

$$f_k(z) = \begin{cases} 1 & \text{if } \operatorname{Im} z > \frac{1}{2k} \\ 0 & \text{if } |\operatorname{Im} z| < \frac{1}{4k} \\ -1 & \text{if } \operatorname{Im} z < \frac{-1}{2k}. \end{cases} \tag{11.1.4.1.2}$$

Then f_k is holomorphic on a neighborhood of K_k. And $\widehat{\mathbb{C}} \setminus K_k$ is connected. So the result in §§11.1.3 applies and there is a holomorphic polynomial p_k

such that

$$\max_{K_k} |p_k(z) - f_k(z)| < 2^{-k}. \tag{11.1.4.1.3}$$

The sequence $\{p_k\}$ has the desired properties. □

EXAMPLE 11.1.4.2 There is a sequence of holomorphic polynomials p_k such that $p_k(0) = 1$ for all k and $p_k \to 0$ pointwise on (indeed uniformly on compact subsets of) $\mathbb{C} \setminus \{0\}$.

Indeed, for $k = 2, 3, \ldots$, let

$$L_k = \overline{D}(0,k) \setminus \left[D\left(0, \frac{1}{k}\right) \cup \left\{ z \in \mathbb{C} : 0 < \operatorname{Im} z < \frac{1}{2k}, \operatorname{Re} z > 0 \right\} \right] \tag{11.1.4.2.1}$$

and

$$K_k = L_k \cup \{0\}. \tag{11.1.4.2.2}$$

See Figure 11.2. Define

$$f_k(z) = \begin{cases} 1 \text{ if } |z| < \frac{1}{4k} \\ 0 \text{ if } |z| > \frac{1}{3k}. \end{cases} \tag{11.1.4.2.3}$$

Then f_k is holomorphic on a neighborhood of K_k, $\widehat{\mathbb{C}} \setminus K_k$ is connected, and the result of §§11.1.3 applies. So there is a polynomial q_k such that

$$\max_{z \in K_k} |q_k(z) - f_k(z)| < 2^{-k}. \tag{11.1.4.2.4}$$

Define

$$p_k(z) = q_k(z) + (1 - q_k(0)). \tag{11.1.4.2.5}$$

Then $p_k(0) = 1$ and

$$\begin{aligned} \max_{K_k} |p_k(z) - f_k(z)| &\leq \max_{K_k} |p_k(z) - q_k(z)| + |q_k(z) - f_k(z)| \\ &= |1 - q_k(0)| + \max_{K_k} |q_k(z) - f_k(z)| \\ &= |f_k(0) - q_k(0)| + \max_{K_k} |q_k(z) - f_k(z)| \\ &< 2^{-k} + 2^{-k} \\ &= 2^{-k+1}. \end{aligned}$$

Thus the sequence $\{p_k\}$ has all the required properties. □

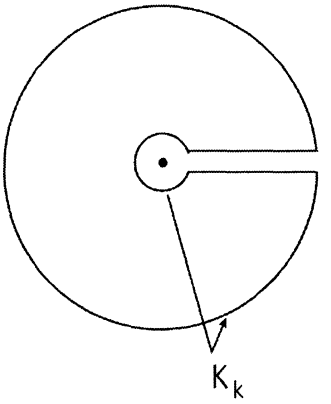

FIGURE 11.2
The region K_k in Example 11.1.4.2.

11.2 Mergelyan's Theorem

11.2.1 An Improvement of Runge's Theorem

Runge's theorem leaves open the question whether conditions **(11.1.1.1)** and **(11.1.1.2)** are sufficient for rational approximation on K. When $\widehat{\mathbb{C}} \setminus K$ has finitely many components, the conditions are sufficient: this is the content of Mergelyan's theorem which we treat in this section. It is startling that a period of sixty-seven years elapsed between the appearance of Runge's theorem (1885) and that of Mergelyan's theorem (1952), for Mergelyan's proof involves no fundamentally new ideas. The proof even uses Runge's theorem.

11.2.2 A Special Case of Mergelyan's Theorem

Let $K \subseteq \mathbb{C}$ be a compact set. We let $C(K)$ denote the space of continuous functions having domain K.

> Let $K \subseteq \mathbb{C}$ be compact and assume that $\widehat{\mathbb{C}} \setminus K$ is connected. Let $f \in C(K)$ be holomorphic on $\overset{\circ}{K}$. Then for any $\epsilon > 0$ there is a holomorphic polynomial $p(z)$ such that
>
> $$\max_{z \in K} |p(z) - f(z)| < \epsilon. \qquad (11.2.2.1)$$

11.2.3 Generalized Mergelyan Theorem

Let $K \subseteq \mathbb{C}$ be compact and suppose that $\widehat{\mathbb{C}} \setminus K$ has only finitely many connected components. If $f \in C(K)$ is holomorphic on $\overset{\circ}{K}$ and if $\epsilon > 0$, then there is a rational function $r(z)$ with poles in $\widehat{\mathbb{C}} \setminus K$ such that

$$\max_{z \in K} |f(z) - r(z)| < \epsilon. \tag{11.2.3.1}$$

Chapter 12

Special Classes of Holomorphic Functions

12.1 Schlicht Functions and the Bieberbach Conjecture

12.1.1 Schlicht Functions

A holomorphic function f on the unit disc D is called *schlicht* if

(12.1.1.1) f is one-to-one,

(12.1.1.2) $f(0) = 0$,

(12.1.1.3) $f'(0) = 1$.

In this circumstance we write $f \in \mathcal{S}$. Such a function has a power series expansion of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j \cdot z^j. \tag{12.1.1.4}$$

If $0 \leq \theta < 2\pi$, then the *Köbe function*

$$f_\theta(z) \equiv \frac{z}{(1 + e^{i\theta} z)^2} \tag{12.1.1.5}$$

is a schlicht function that satisfies $|a_j| = j$ for all j.

12.1.2 The Bieberbach Conjecture

It was a problem of long standing to show that the Köbe functions are the only schlicht functions whose coefficients have this property. It was

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further conjectured that for any schlicht f it holds that $|a_j| \leq j$. This last was known as the *Bieberbach conjecture*. Both of these problems were solved in the affirmative by Louis De Branges [DBR] in 1984.

In this section we list briefly some of the ingredients that have been used historically to understand the Bieberbach conjecture.

12.1.3 The Lusin Area Integral

Let $\Omega \subseteq \mathbb{C}$ be a domain and $\varphi : \Omega \to \mathbb{C}$ a one-to-one holomorphic function. Then $\varphi(\Omega)$ is a domain and

$$\text{area}(\varphi(\Omega)) = \int_{\Omega} |\varphi'(z)|^2 \, dxdy. \tag{12.1.3.1}$$

12.1.4 The Area Principle

The Area Principle If f is schlicht and if

$$h(z) = \frac{1}{f(z)} = \frac{1}{z} + \sum_{j=0}^{\infty} b_j z^j \,, \tag{12.1.4.1}$$

then

$$\sum_{j=1}^{\infty} j|b_j|^2 \leq 1. \tag{12.1.4.2}$$

12.1.5 The Köbe 1/4 Theorem

If f is schlicht, then

$$f(D(0,1)) \supseteq D(0,1/4). \tag{12.1.5.1}$$

Remark: Note that if $f \in \mathcal{S}$, then $f(0) = 0$ and $f'(0) = 1 \neq 0$, hence, by the Inverse Function Theorem (see [RUD1]), the image of f will contain an open neighborhood of the origin. A similar conclusion may be derived from the open mapping principle of complex variable theory. Köbe's theorem estimates the size of that neighborhood.

Remark: Historically, (12.1.4.2) and (12.1.5.1) were of interest in schlicht function theory because they were used to estimate the size of $|a_2|$. By separate, and increasingly difficult, arguments the estimates $|a_3| \leq 3$, $|a_4| \leq 4$, $|a_5| \leq 5$, $|a_6| \leq 6$, and so on up to $|a_8|$ were successfully proved. Some of the history is as follows:

1916	Bieberbach	$\lvert a_2\rvert \leq 2$
1923	Loewner	$\lvert a_3\rvert \leq 3$
1955	Garabedian & Schiffer	$\lvert a_4\rvert \leq 4$
1968	Ozawa & Pederson	$\lvert a_6\rvert \leq 6$
1972	Pederson and Schiffer	$\lvert a_5\rvert \leq 5$

The full Bieberbach conjecture was proved in 1984. The complete history prior to 1984 is given in the book [DUR].

12.2 Extension to the Boundary of Conformal Mappings

12.2.1 Boundary Continuation

Let Ω_1, Ω_2 be domains in $\mathbb{C}$ and let $\varphi : \Omega_1 \to \Omega_2$ be a conformal (one-to-one, onto, holomorphic) mapping. It is of both aesthetic and practical interest to consider whether φ extends continuously to $\partial\Omega_1$ (and, likewise, φ^{-1} to $\partial\Omega_2$). One might expect that, the nicer the boundaries of Ω_1 and Ω_2, the more likely the existence of a continuous extension.

It turns out that a continuous extension to the boundary exists virtually all the time, and the extension is more regular when the boundary is more regular. Here are two extreme examples:

12.2.2 Some Examples Concerning Boundary Continuation

EXAMPLE 12.2.2.1 The conformal mapping

$$\varphi : D(0,1) \to \{z \in D(0,1) : \operatorname{Im} z > 0\} \tag{12.2.2.1.1}$$

given by

$$\varphi(z) = \frac{1 + e^{i(\pi/4)}\left([1-z]/[1+z]\right)^{3/2}}{1 - e^{i(\pi/4)}\left([1-z]/[1+z]\right)^{3/2}} \tag{12.2.2.1.2}$$

extends continuously but not differentiably to ± 1. Of course the boundary of $\{z \in D(0,1) : \operatorname{Im} z > 0\}$ is only piecewise differentiable: it has corners at ± 1. □

EXAMPLE 12.2.2.2 Any conformal map of the disc to the disc has the

form

$$z \to e^{i\theta_0} \cdot \frac{z-a}{1-\overline{a}z}, \tag{12.2.2.2.1}$$

some $\theta_0 \in [0, 2\pi)$ and $a \in D$ (see §§6.2.3). Such a map continues holomorphically past every point of ∂D. Note that ∂D is real analytic. □

These two examples are indicative of some general phenomena that occur. We will now describe three of the most basic results. Let Ω_1, Ω_2 be bounded domains and $\varphi : \Omega_1 \to \Omega_2$ be a conformal mapping.

(12.2.2.3) Carathéodory's theorem: If $\partial\Omega_1, \partial\Omega_2$ are Jordan curves (simple, closed curves), then φ (resp. φ^{-1}) extends one-to-one and continuously to $\partial\Omega_1$ (resp. $\partial\Omega_2$).

(12.2.2.4) If $\partial\Omega_1, \partial\Omega_2$ are C^∞, then φ (resp. φ^{-1}) extends one-to-one to $\partial\Omega_1$ (resp. $\partial\Omega_2$) in such a fashion that the extension is C^∞ on $\overline{\Omega}_1$ (resp. $\overline{\Omega}_2$).

(12.2.2.5) If $\partial\Omega_1, \partial\Omega_2$ are real analytic, then φ (resp. φ^{-1}) continues holomorphically past $\partial\Omega_1$ (resp. $\partial\Omega_2$).

It should be noted that these results do not hold as stated if one of the domains is unbounded. Consider, for instance, the Cayley transform $\varphi : D \to \{z : \operatorname{Im} z > 0\}$ given by $\varphi(z) = i(1-z)/(1+z)$.

12.3 Hardy Spaces

12.3.1 The Definition of Hardy Spaces

If $0 < p < \infty$, then we define $H^p(D)$ to be the class of those functions holomorphic on the disc and satisfying the growth condition

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty. \tag{12.3.1.1}$$

In this circumstance we write $f \in H^p(D)$. It is convenient to use the notation $\|f\|_{H^p}$ to denote the quantity displayed in (12.3.1.1).

We also let $H^\infty(D)$ denote the class of bounded holomorphic functions on D, and we let $\|f\|_{H^\infty}$ denote the supremum of f on D. (The notation originates from the use of L^∞ for bounded functions, with the essential supremum norm, in real analysis—see [RUD2], [FOL].)

12.3.2 The Blaschke Factorization for H^∞

In §9.1 we studied the H^∞ functions and characterized their zero sets. We also discovered (see §§9.1.7) that every H^∞ function f has a factorization

$$f(z) = F(z) \cdot B(z) \tag{12.3.2.1}$$

where $B(z)$ is a Blaschke product having the same zeros (counting multiplicities) as f and F is a non-vanishing holomorphic function such that $\|F\|_{H^\infty} = \|f\|_{H^\infty}$.

It is clear that for any p we have $H^p(D) \supseteq H^\infty(D)$. G. H. Hardy, M. Riesz, and others, in a series of papers published in the 1920s, discovered that $H^p(D)$ functions share many of the properties of $H^\infty(D)$ functions. In particular, it turns out that the zero sets of $H^p(D)$ functions are just the same as those for $H^\infty(D)$ functions, and that a factorization such as (12.3.2.1) is valid. We shall investigate these matters in the present section.

First we state a convenient result which tells us, in effect, that $\|f\|_{H^p}$ is attained as $r \to 1^-$.

12.3.3 Monotonicity of the Hardy Space Norm

Let f be holomorphic on D. If $0 < r_1 < r_2 < 1$, then

$$\int_0^{2\pi} |f(r_1 e^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |f(r_2 e^{i\theta})|^p \, d\theta. \tag{12.3.3.1}$$

12.3.4 Containment Relations among Hardy Spaces

If $0 < p_1 < p_2 < \infty$, then

$$H^{p_2}(D) \underset{\neq}{\subset} H^{p_1}(D). \tag{12.3.4.1}$$

EXAMPLE 12.3.4.2 Let $m = [p_1 + p_2]/2$. The function $f(z) = (1 - z)^{-1/m}$ lies in H^{p_1} but not in H^{p_2}. These assertions can be verified with elementary integrations. □

12.3.5 The Zeros of Hardy Functions

If $f \in H^p, 0 < p \leq \infty$, and if $a_1, a_2, \ldots$ are the zeros of f (listed with multiplicities), then $\sum_j (1 - |a_j|) < \infty$.

These facts lead to a factorization result for H^p functions:

12.3.6 The Blaschke Factorization for H^p Functions

Let $0 < p \leq \infty$. If $f \in H^p(D)$ and if f vanishes to order m at 0, then there is a convergent Blaschke product B and a non-vanishing holomorphic F on D such that

$$f(z) = z^m \cdot F(z) \cdot B(z). \tag{12.3.6.1}$$

Furthermore, $\|F\|_{H^p} = \|f\|_{H^p}$.

Chapter 13

Special Functions

13.0 Introduction

While the spirit of modern analysis is to study classes or types of functions, there are some particular functions that arise so frequently that they have intrinsic interest. These functions are usually termed *special functions.* In this chapter we shall treat three of these which arise naturally in complex analysis: the gamma function of Euler, the beta function of Legendre, and the ζ (or zeta) function of Riemann.

13.1 The Gamma and Beta Functions

13.1.1 Definition of the Gamma Function

If $\operatorname{Re} z > 0$, then define

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{13.1.1.1}$$

We have already seen this function in §§10.1.2.

Note that

$$|t^{z-1}e^{-t}| \leq t^{\operatorname{Re} z - 1}, \tag{13.1.1.2}$$

so that the integral converges at $t = 0$. Likewise, for t large,

$$|t^{z-1}e^{-t}| = t^{\operatorname{Re} z - 1}e^{-t} \leq e^{t/2}e^{-t} = e^{-t/2}, \tag{13.1.1.3}$$

so that the integral converges at ∞. Thus Γ is a well-defined function for z in the right half plane. The following subsections present formally a set of ideas that we developed briefly in Example 10.1.2.2 to illustrate the notion of analytic continuation.

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13.1.2 Recursive Identity for the Gamma Function

A simple calculation shows that $\Gamma(1) = 1$. If $\operatorname{Re} z > 0$, then

$$\Gamma(z+1) = z \cdot \Gamma(z). \tag{13.1.2.1}$$

For $n = 1, 2, \ldots,$

$$\Gamma(n) = (n-1)! \tag{13.1.2.2}$$

13.1.3 Holomorphicity of the Gamma Function

We can use Morera's theorem (§§2.3.2) to see that the function $\Gamma(z)$ is holomorphic on the right half plane.

13.1.4 Analytic Continuation of the Gamma Function

The function Γ continues as a meromorphic function to all of $\mathbb{C}$. The only poles are simple poles at $0, -1, -2, \ldots$. The residue at $-k$ is $(-1)^k/k!$.

Note that, by the uniqueness of analytic continuation, the functional equation $\Gamma(z+1) = z \cdot \Gamma(z)$ now holds for all $z \notin \{0, -1, -2, \ldots\}$. Thus the Γ function provides a natural meromorphic continuation to $\mathbb{C}$ of the factorial function on the positive integers.

13.1.5 Product Formula for the Gamma Function

If $z \notin \{0, -1, -2, \ldots\}$, then

$$\begin{aligned}\Gamma(z) &= \frac{1}{z} \prod_{j=1}^{\infty} \left[\left(1 + \frac{1}{j}\right)^z \left(1 + \frac{z}{j}\right)^{-1} \right] \\ &= \lim_{n \to \infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.\end{aligned} \tag{13.1.5.1}$$

13.1.6 Non-Vanishing of the Gamma Function

The Γ function never vanishes.

13.1.7 The Euler-Mascheroni Constant

The limit

$$\lim_{n \to \infty} \left\{ \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \log n \right\} \tag{13.1.7.1}$$

exists. The limit is a positive constant denoted by γ and called the Euler-Mascheroni constant.

Calculations show that

$$\gamma = 0.57721566\ldots \tag{13.1.7.2}$$

to an accuracy of eight decimal places. It is unknown whether γ is rational or irrational.

13.1.8 Formula for the Reciprocal of the Gamma Function

For any $z \in \mathbb{C}$,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{j=1}^{\infty} \left[\left(1 + \frac{z}{j}\right) e^{-z/j} \right]. \tag{13.1.8.1}$$

Weierstrass gave (13.1.8.1) as the definition of the gamma function.

13.1.9 Convexity of the Gamma Function

For $x > 0$, $\log \Gamma(x)$ is a convex function.

13.1.10 The Bohr-Mollerup Theorem

Suppose that $\varphi : (0, \infty) \to (0, \infty)$ satisfies

(13.1.10.1) $\log \varphi(x)$ is convex;

(13.1.10.2) $\varphi(x+1) = x \cdot \varphi(x)$, all $x > 0$;

(13.1.10.3) $\varphi(1) = 1$.

Then $\varphi(x) \equiv \Gamma(x)$. Thus (by analytic continuation) Γ is the only meromorphic function on $\mathbb{C}$ satisfying the functional equation $z\Gamma(z) = \Gamma(z+1), \Gamma(1) = 1$, and which is logarithmically convex on the positive real axis.

13.1.11 The Beta Function

An auxiliary special function that is closely related to the gamma function is the beta function

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt, \tag{13.1.11.1}$$

which is defined by this formula for $\operatorname{Re} z > 0, \operatorname{Re} w > 0$.

13.1.12 Symmetry of the Beta Function

For any z, w with positive real parts,

$$B(z,w) = B(w,z). \tag{13.1.12.1}$$

13.1.13 Relation of the Beta Function to the Gamma Function

For $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$,

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{13.1.13.1}$$

Note that this last identity provides an analytic continuation of B to all z, w such that $z, w \notin \{0, -1, -2, \ldots\}$.

13.1.14 Integral Representation of the Beta Function

For $z, w \notin \{0, -1, -2, \ldots\}$,

$$B(z,w) = 2\int_0^{\pi/2} (\sin\theta)^{2z-1}(\cos\theta)^{2w-1}\, d\theta. \tag{13.1.14.1}$$

13.2 Riemann's Zeta Function

13.2.1 Definition of the Zeta Function

For $\operatorname{Re} z > 1$, define

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z\log n}. \tag{13.2.1.1}$$

Clearly the series converges absolutely and uniformly on compact subsets of $\{z : \operatorname{Re} z > 1\}$, since $|n^z| = n^{\operatorname{Re} z}$. So ζ is a holomorphic function on $\{z : \operatorname{Re} z > 1\}$. Our primary task in this section is to learn how to continue the zeta function analytically to be a meromorphic function on $\mathbb{C} \setminus \{1\}$ and to learn something about the location of its zeros.

13.2.2 The Euler Product Formula

For motivational purposes, let us immediately establish the connection between ζ and the collection of prime numbers.

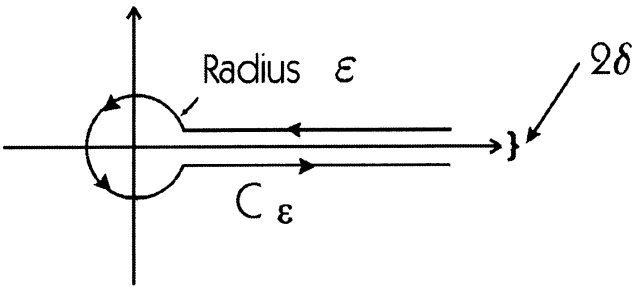

FIGURE 13.1
The Hankel contour.

Let $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\} = \{p_1, p_2, \ldots\}$ be the set of positive primes. For $\operatorname{Re} z > 1$, the infinite product $\prod_{p \in \mathcal{P}}(1 - 1/p^z)$ converges and

$$\frac{1}{\zeta(z)} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^z}\right). \tag{13.2.2.1}$$

13.2.3 Relation of the Zeta Function to the Gamma Function

For $\operatorname{Re} z > 1$,

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-t}}{1 - e^{-t}}\, dt. \tag{13.2.3.1}$$

13.2.4 The Hankel Contour and Hankel Functions

For $z \in \mathbb{C}$ fixed, define the auxiliary function

$$u : w \mapsto \frac{(-w)^{z-1} e^{-w}}{1 - e^{-w}}. \tag{13.2.4.1}$$

On the region $\mathbb{C} \setminus \{w : \operatorname{Re} w \geq 0\}$, this function is well-defined if we take $-\pi < \arg(-w) < \pi$. Also define, for $0 < \epsilon \neq 2k\pi$, what are called the Hankel functions

$$H_\epsilon(z) = \int_{C_\epsilon} u(w)\, dw, \tag{13.2.4.2}$$

where $C_\epsilon = C_\epsilon(\delta)$ is the "Hankel contour" shown in Figure 13.1. Note that the contour is given the usual positive orientation. The linear portions of the contour are understood to lie just above and just below the real axis (at distance $\delta > 0$ as shown). The circular portion is understood to have radius ϵ.

Note that, for $0 < \epsilon_1 < \epsilon_2 < 2\pi$, $H_{\epsilon_1}(z) = H_{\epsilon_2}(z)$ since the region (see Figure 13.2) bounded by $C_{\epsilon_1}, C_{\epsilon_2}$ contains no poles of u.

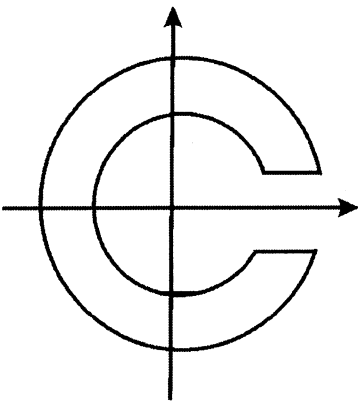

FIGURE 13.2
The region bounded by $C_{\epsilon_1}, C_{\epsilon_2}$ contains no poles of u.

13.2.5 Expression of the Zeta Function as a Hankel Integral

For $0 < \epsilon < 2\pi$ and $\operatorname{Re} z > 1$ we have

$$\zeta(z) = \frac{-H_\epsilon(z)}{2i \sin(\pi z)\Gamma(z)}. \tag{13.2.5.1}$$

As a result, ζ continues analytically to $\mathbb{C} \setminus \{1\}$.

13.2.6 Location of the Pole of the Zeta Function

The zeta function has a simple pole at $z = 1$ with residue 1. It has no other pole.

13.2.7 The Functional Equation

The following fact (the functional equation) provides more explicit information about the continuation of ζ to $\mathbb{C} \setminus \{1\}$.

For all $z \in \mathbb{C}$,

$$\zeta(1 - z) = 2\zeta(z)\Gamma(z) \cos\left(\frac{\pi}{2} z\right) \cdot (2\pi)^{-z}. \tag{13.2.7.1}$$

Observe that the functional equation gives us an explicit way to extend the definition of the zeta function to all of $\mathbb{C} \setminus \{1\}$, whereas before we only knew (§§13.2.5) that such a continuation existed. The functional equation is a sort of reflection formula, giving the values to the left of $\operatorname{Re} z = 1$ in terms of those to the right.

13.2.8 Zeros of the Zeta Function

The Euler product formula tells us that, for $\operatorname{Re} z > 1, \zeta(z)$ does not vanish. Since Γ never vanishes, the functional equation thus says that $\zeta(1-z)$ can vanish for $\operatorname{Re} z > 1$ only at the zeros of $\cos((\pi/2)z)$, that is only at $z = 3, 5, \ldots$ Thus we can establish the following.

The only zeros of ζ not in the set $\{z : 0 \leq \operatorname{Re} z \leq 1\}$ are at $-2, -4, -6, \ldots$.

13.2.9 The Riemann Hypothesis

Since all the zeros of ζ except those at $-2, -4, -6, \ldots$ are in the strip $\{z : 0 \leq \operatorname{Re} z \leq 1\}$, questions about the zeros of ζ focus attention on this strip. People have consequently, over the years, taken to calling this the *critical strip.*

It is known that the zeta function does not vanish on $\{z : \operatorname{Re} z = 1\}$ nor on $\{z : \operatorname{Re} z = 0\}$ and that it has infinitely many zeros in the interior of the critical strip (this last is a theorem of Hardy). The celebrated Riemann Hypothesis is the conjecture that all the zeros of ζ in the critical strip actually lie on the line $\{z : \operatorname{Re} z = 1/2\}$. The interested reader can refer to [EDW] for further information about this important unsolved problem. In this section we shall only discuss the non-vanishing of ζ on the boundary of the strip. In order to achieve this goal, we must bring the prime numbers $\mathcal{P}$ into play.

13.2.10 The Lambda Function

Define the function

$$\Lambda : \{n \in \mathbb{Z} : n > 0\} \to \mathbb{R} \tag{13.2.10.1}$$

by the condition

$$\Lambda(m) = \begin{cases} \log p \text{ if } m = p^k, \, p \in P, \, 0 < k \in \mathbb{Z} \\ 0 \qquad \text{otherwise.} \end{cases} \tag{13.2.10.2}$$

13.2.11 Relation of the Zeta Function to the Lambda Function

For $\operatorname{Re} z > 1$,

$$\sum_{n \geq 2} \Lambda(n) e^{-z \log n} = \frac{-\zeta'(z)}{\zeta(z)}. \tag{13.2.11.1}$$

Equation (13.2.11.1) tells us about the behavior of ζ *to the right* of $\{z : \operatorname{Re} z = 1\}$ while we actually need information *on the line* $\{z : \operatorname{Re} z = 1\}$. The next result, in contrapositive form, is our device for obtaining this information.

13.2.12 More on the Zeros of the Zeta Function

If φ is holomorphic in a neighborhood of $P \in \mathbb{R}$, if φ not identically zero, and if $\varphi(P) = 0$, then

$$\operatorname{Re} \frac{\varphi'(z)}{\varphi(z)} > 0 \tag{13.2.12.1}$$

for $z \in \mathbb{R}$ near P and to the right of P. This information is important when the location of the zeros of ζ is being studied.

13.2.13 Zeros of the Zeta Function and the Boundary of the Critical Strip

The Riemann zeta function has no zero on the boundary of the critical strip.

13.3 Some Counting Functions and a Few Technical Lemmas

13.3.1 The Counting Functions of Classical Number Theory

In this section we define one of the counting functions of classical number theory. Traditionally, it is studied by way of Riemann's zeta function.

13.3.2 The Function π

If $x > 0$, we let

$$\pi(x) = \text{the number of prime numbers not exceeding } x. \tag{13.3.2.1}$$

The prime number theorem is a statement about the growth of $\pi(x)$ as $x \to \infty$.

13.3.3 The Prime Number Theorem

The prime number theorem states that

$$\lim_{x \to \infty} \frac{\pi(x)}{(x/\log x)} = 1. \tag{13.3.3.1}$$

Chapter 14

Applications that Depend on Conformal Mapping

14.1 Conformal Mapping

14.1.1 A List of Useful Conformal Mappings

Part of the utility of conformal mappings is that they can be used to transform a problem on a given domain V to another domain U (see also §§6.1.1). Often we take U to be a standard domain such as the disc

$$D = \{z \in \mathbb{C} : |z| < 1\} \tag{14.1.1.1}$$

or the upper half plane

$$U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}. \tag{14.1.1.2}$$

Particularly in the study of partial differential equations, it is important to have an *explicit* conformal mapping between the two domains. In the Appendix to this chapter we give a compendium of conformal mappings of some of the most frequently encountered planar regions.

The reader will find that, even in cases where the precise mapping that he seeks has not been listed, he can (much as with a table of integrals) combine several of the given mappings to produce the results that he seeks. It is also the case that the techniques presented can be modified to suit a variety of different situations.

The references [KOB] and [CCP] give more comprehensive lists of conformal mappings.

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14.2 Application of Conformal Mapping to the Dirichlet Problem

14.2.1 The Dirichlet Problem

Let $\Omega \subseteq \mathbb{C}$ be a domain whose boundary consists of finitely many smooth curves. The *Dirichlet problem* (see §7.3, §7.7, §7.8), which is a mathematical problem of interest in its own right, is the boundary value problem

$$\begin{aligned} \triangle u &= 0 \quad \text{on } \Omega \\ u &= f \quad \text{on } \partial\Omega. \end{aligned} \tag{14.2.1.1}$$

The way to think about this problem is as follows: a data function f on the boundary of the domain is given. To solve the corresponding Dirichlet problem, one seeks a harmonic function u on the closure of U (i.e., the union of U and its boundary) such that u agrees with f on the boundary. We shall now describe three distinct physical situations that are mathematically modeled by the Dirichlet problem.

14.2.2 Physical Motivation for the Dirichlet Problem

I. Heat Diffusion: Imagine that Ω is a thin plate of heat-conducting metal. The shape of Ω is arbitrary (not necessarily a rectangle). See Figure 14.1. A function $u(x, y)$ describes the temperature at each point (x, y) in Ω. It is a standard situation in engineering or physics to consider idealized heat sources or sinks that maintain specified (fixed) values of u on certain parts of the boundary; other parts of the boundary are to be thermally insulated. One wants to find the steady state heat distribution on Ω that is determined by the given boundary conditions. If we let f denote the temperature specified on the boundary, then it turns out the solution of the Dirichlet problem (14.2.1.1) is the function that describes the heat distribution (see [COH], [KRA4], [BCH, p.300] and references therein for a derivation of this mathematical model for heat distribution).

We will present below some specific examples of heat diffusion problems that illustrate the mathematical model that we have discussed here, and we will show how conformal mapping can be used in aid of the solutions of the problems.

II. Electrostatic Potential: Now we describe a situation in electrostatics that is modeled by the boundary value problem (14.2.1.1).

Imagine a long, hollow cylinder made of a thin sheet of some conducting material, such as copper. Split the cylinder lengthwise into two equal pieces (Figure 14.2). Separate the two pieces with strips of insulating material

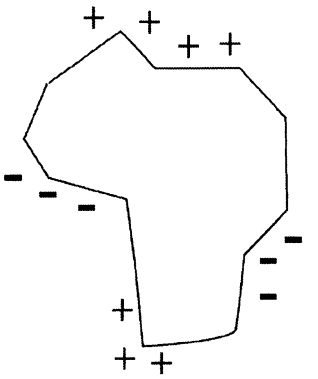


FIGURE 14.1
Heat distribution on the edge of a metal plate.

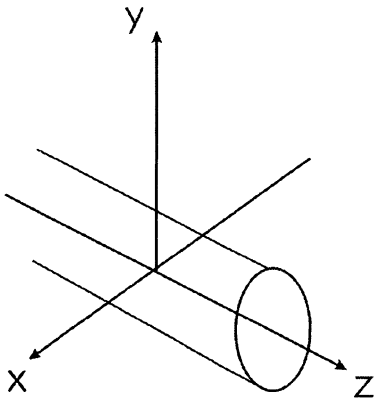


FIGURE 14.2
Electrostatic potential illustrated with a split cylinder.

(Figure 14.3). Now ground the upper of the two semi-cylindrical pieces to potential zero, and keep the lower piece at some non-zero fixed potential. For simplicity in the present discussion, let us say that this last fixed potential is 1.

Note that, in the figures, the axis of the cylinder is the z-axis. Consider a slice of this cylindrical picture which is taken by setting z equal to a small constant (we want to stay away from the ends of the cylinder, where the analysis will be a bit different).

Once we have fixed a value of z, then we may study the electrostatic potential $V(x, y)$, $x^2 + y^2 < 1$, at a point inside the cylinder. Observe that $V = 0$ on the "upper" half of the circle ($y > 0$) and $V = 1$ on the "lower" half of the circle ($y < 0$)—see Figure 14.4. Physical analysis (see [BCH,

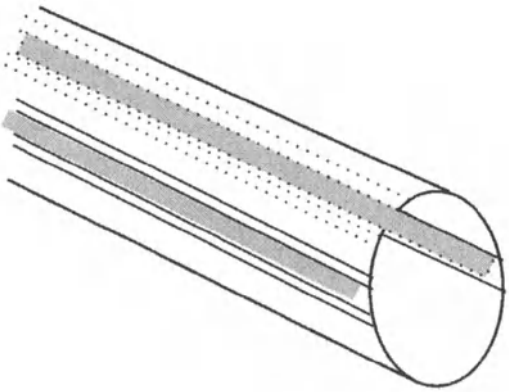


FIGURE 14.3
The cylindrical halves separated with insulating material.

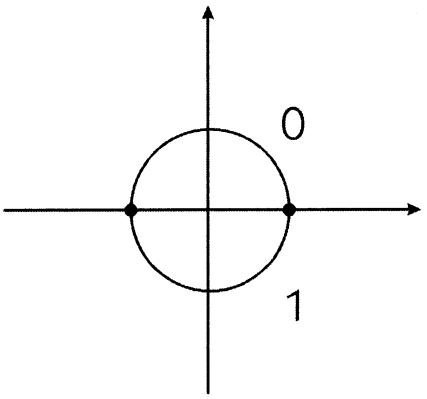


FIGURE 14.4
Distribution of the electrostatic potential.

p. 310]) shows that this is another Dirichlet problem, as in (14.2.1.1). We wish to find a harmonic function V on the disc $\{(x, y) : x^2 + y^2 \leq 1\}$ which agrees with the given potentials on the boundary.

Conformal mapping can be used as an aid in solving the problem posed here, and we shall discuss its solution below.

III. Incompressible Fluid Flow: For the mathematical model considered here, we consider a two-dimensional flow of a fluid that is

- incompressible
- irrotational
- free from viscosity

The first of these stipulations means that the fluid is of constant density, the second means that the curl is zero, and the third means that the fluid flows freely.

Identifying a point (a, b) in the x - y plane with the complex number $a + ib$ as usual, we let

$$V(x, y) = p(x, y) + iq(x, y) \tag{14.2.2.1}$$

represent the velocity vector of our fluid flow at a point (x, y). We assume that the fluid flow has no sources or sinks; and we hypothesize that p and q are C^1, or once continuously differentiable (see §§1.3.1).

The *circulation* of the fluid along any curve γ is the line integral

$$\int_\gamma V_T(x, y)\, d\sigma. \tag{14.2.2.2}$$

Here V_T represents the tangential component of the velocity along the curve γ and σ denotes arc length. We know from advanced calculus that the circulation can be written

$$\int_\gamma p(x, y)\, dx + \int_\gamma q(x, y)\, dy. \tag{14.2.2.3}$$

We assume here that γ is a positively (counterclockwise) oriented simple, closed curve that lies in a simply connected region D of the flow.

Now Green's theorem allows us to rewrite this last expression for the circulation as

$$\iint_R [q_x(x, y) - p_y(x, y)]\, dA. \tag{14.2.2.4}$$

Here the subscripts x and y represent partial derivatives, R is the region surrounded by γ, and dA is the element of area. In summary

$$\int_\gamma V_T(x, y)\, d\sigma = \iint_R [q_x(x, y) - p_y(x, y)]\, dA. \tag{14.2.2.5}$$

Let us specialize to the case that γ is a circle of radius r with center $p_0 = (x_0, y_0)$. Then the mean angular speed of the flow along γ is

$$\frac{1}{\pi r^2} \iint_R \frac{1}{2}[q_x(x, y) - p_y(x, y)]\, dA. \tag{14.2.2.6}$$

This expression also happens to represent the average of the function

$$\omega(x, y) = \frac{1}{2}[q_x(x, y) - p_y(x, y)] \tag{14.2.2.7}$$

over R. Since ω is continuous, the limit as $r \to 0$ of (14.2.2.6) is just $\omega(p_0)$. It is appropriate to call ω the *rotation* of the fluid, since it is the limit at the point p_0 of the angular speed of a circular element of the fluid at the point p_0. Since our fluid is irrotational, we set $\omega = 0$. Thus we know that

$$p_y = q_x \tag{14.2.2.8}$$

in the region D where the flow takes place. Multi-dimensional calculus then tells us that the flow is path-independent: If $X = (x, y)$ is any point in the region and γ is any path joining p_0 to X, then the integral

$$\int_\gamma p(s,t)\,ds + \int_\gamma q(s,t)\,dt \tag{14.2.2.9}$$

is independent of the choice of γ. As a result, the function

$$\varphi(x,y) = \int_{p_0}^{X} p(s,t)\,ds + q(s,t)\,dt \tag{14.2.2.10}$$

is well-defined on D. Differentiating the equation that defines φ, we find that

$$\frac{\partial}{\partial x}\varphi(x,y) = p(x,y) \qquad \text{and} \qquad \frac{\partial}{\partial y}\varphi(x,y) = q(x,y). \tag{14.2.2.11}$$

We call φ a *potential function* for the flow. To summarize, we know that $\nabla\varphi = (p, q)$.

The natural physical requirement that the incompressible fluid enter or leave an element of volume only by flowing through the boundary of that element (no sources or sinks) entails the mathematical condition that φ be harmonic. Thus

$$\varphi_{xx} + \varphi_{yy} = 0 \tag{14.2.2.12}$$

on D. In conclusion, studying a fluid flow with specified boundary data will entail solving the boundary value problem (14.2.1.1).

14.3 Physical Examples Solved by Means of Conformal Mapping

In this section we give a concrete illustration of the solution of each of the physical problems described in the last section.

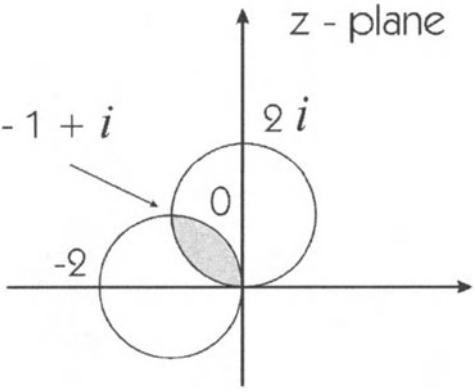


FIGURE 14.5
A lens-shaped piece of heat-conducting metal.

14.3.1 Steady State Heat Distribution on a Lens-Shaped Region

EXAMPLE 14.3.1.1 Imagine a lens-shaped piece of heat-conducting metal as in Figure 14.5. Suppose that the initial distribution of heat is specified to be 1 on the lower boundary of the lens and 0 on the upper boundary of the lens (as illustrated in the figure). Determine the steady state heat distribution.

Solution: Our strategy is to use a conformal mapping to transfer the problem to a new domain on which it is easier to work. We let $z = x + iy$ denote the variable in the lens-shaped region and $w = u + iv$ denote the variable in the new region (which will be an angular region).

In fact let us construct the conformal mapping with our bare hands. If we arrange for the mapping to be linear fractional and to send the origin to the origin and the point $-1 + i$ to infinity, then (since linear fractional transformations send lines and circles to lines and circles), the images of the two circular arcs will both be lines. Let us in fact examine the mapping

$$w = f(z) = \frac{-z}{z - (-1 + i)}. \tag{14.3.1.1.1}$$

(The minus sign in the numerator is introduced for convenience.)

We see that $f(0) = 0$, $f(2i) = -1 - i$, and $f(-2) = -1 + i$. Using conformality (preservation of right angles), we conclude that the image of the lens-shaped region in the z-plane is the angular region in the w-plane depicted in Figure 14.6. The figure also shows on which part of the boundary the function we seek is to have value 0 and on which part it is to have value 1. It is easy to write down a harmonic function of the w

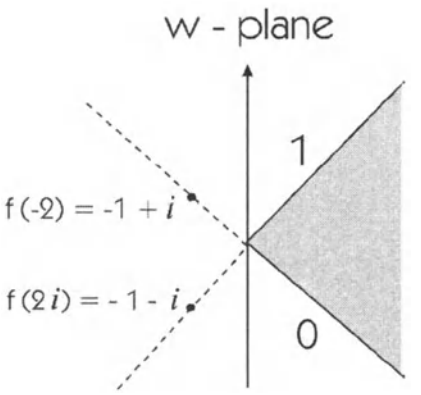


FIGURE 14.6
The angular region in the w-plane that is the image of the lens-shaped region in the z-plane.

variable that satisfies the required boundary conditions:

$$\varphi(w) = \frac{2}{\pi}\left(\arg w + \frac{\pi}{4}\right) \tag{14.3.1.1.2}$$

will certainly do the job if we demand that $-\pi < \arg w < \pi$. But then the function

$$u(z) = \varphi \circ f(z) \tag{14.3.1.1.3}$$

is a harmonic function on the lens-shaped domain in the z-plane that has the requisite boundary values (we use of course the fact that the composition of a harmonic function with a holomorphic function is still harmonic).

In other words, the solution to the problem originally posed on the lens-shaped domain in the z-plane is

$$u(z) = \frac{2}{\pi} \cdot \arg\left(\frac{-z}{z-(-1+i)}\right) + \frac{1}{2}. \tag{14.3.1.1.4}$$

This can also be written as

$$u(z) = \frac{2}{\pi} \cdot \tan^{-1}\left[\frac{-y-x}{-x(x+1)-y(y-1)}\right] + \frac{1}{2}. \tag{14.3.1.1.5}$$

□

14.3.2 Electrostatics on a Disc

EXAMPLE 14.3.2.1 We now analyze the problem that was set up in part II of §§14.2.2. We do so by conformally mapping the unit disc (in the

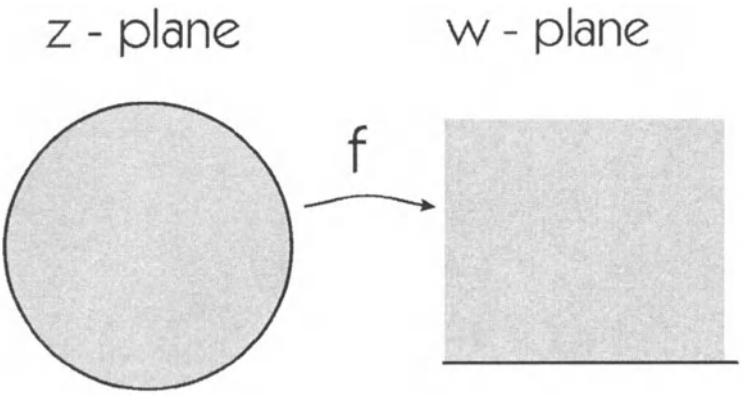


FIGURE 14.7
Conformal map of the disc to the upper half-plane.

z plane) to the upper half plane (in the w plane) by way of the mapping

$$w = f(z) = i \cdot \frac{1-z}{1+z}. \tag{14.3.2.1.1}$$

See Figure 14.7. Observe that this conformal mapping takes the upper half of the unit circle to the positive real axis, the lower half of the unit circle to the negative real axis, and the point 1 to the origin.

Thus we are led to consider the following boundary value problem on the upper half plane in the w variable: we seek a harmonic function on the upper half plane with boundary value 0 on the positive real axis and boundary value 1 on the negative real axis. Certainly the function

$$\varphi(w) = \frac{1}{\pi} \arg w \tag{14.3.2.1.2}$$

does the job, if we assume that $0 \leq \arg w < 2\pi$. We pull this solution back to the disc by way of the mapping f:

$$u = \varphi \circ f. \tag{14.3.2.1.3}$$

This function u is harmonic on the unit disc, has boundary value 0 on the upper half of the circle, and boundary value 1 on the lower half of the circle.

Our solution may be written more explicitly as

$$u(z) = \frac{1}{\pi} \arg \left[i \cdot \frac{1-z}{1+z} \right] \tag{14.3.2.1.4}$$

or as

$$u(z) = \frac{1}{\pi} \tan^{-1} \left[\frac{1-x^2-y^2}{2y} \right]. \tag{14.3.2.1.5}$$

Of course for this last form of the solution to make sense, we must take $0 \leq \arctan t \leq \pi$ and we must note that

$$\lim_{\substack{t \to 0 \\ t>0}} \arctan t = 0 \qquad \text{and} \qquad \lim_{\substack{t \to 0 \\ t<0}} \arctan t = \pi. \tag{14.3.2.1.6}$$

□

14.3.3 Incompressible Fluid Flow around a Post

EXAMPLE 14.3.3.1 We study the classic problem of the flow of an incompressible fluid around a cylindrical post.

Recall the potential function φ from the end of the discussion in III of §§14.2.2. If

$$V = p + iq \tag{14.3.3.1.1}$$

is the velocity vector, then we may write

$$V = \varphi_x + i\varphi_y = \operatorname{grad} \varphi. \tag{14.3.3.1.2}$$

Since φ is harmonic, we may select a conjugate harmonic function ψ (see §§1.4.2) for φ. Because of the Cauchy-Riemann equations, the velocity vector will be tangent to any curve $\psi(x, y) =$ constant. The function ψ is called the *stream function* for the flow. The curves $\psi(x, y) =$ constant are called *streamlines* of the fluid flow. We call the holomorphic function

$$H(x + iy) = \varphi(x, y) + i\psi(x, y) \tag{14.3.3.1.3}$$

the *complex potential* of the fluid flow.

Using the Cauchy-Riemann equations twice, we can write $H'(z)$ as

$$H'(z) = \varphi_x(x, y) + i\psi_x(x, y) \tag{14.3.3.1.4}$$

or

$$H'(z) = \varphi_x(x, y) - i\varphi_y(x, y). \tag{14.3.3.1.5}$$

Thus formula (14.3.3.1.2) for the velocity becomes

$$V = \overline{H'(z)}. \tag{14.3.3.1.6}$$

As a result,

$$\text{speed} = |V| = |\overline{H'(z)}| = |H'(z)|. \tag{14.3.3.1.7}$$

The analysis we have just described means that, in order to solve an incompressible fluid flow problem, we need to find the complex potential function H.

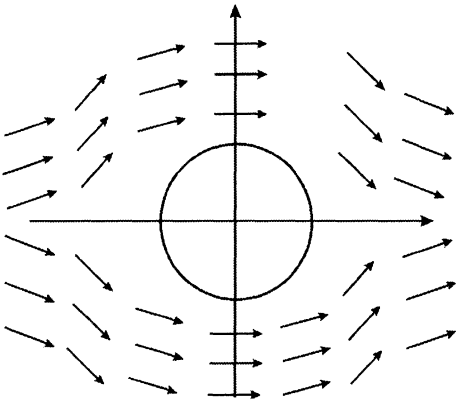


FIGURE 14.8
Incompressible fluid flow with a circular obstacle.

Now consider an incompressible fluid flow with a circular obstacle as depicted in Figure 14.8. The flow is from left to right. Far away from the obstacle, the flow is very nearly along horizontal lines parallel to the x-axis. But near to the obstacle the flow will be diverted. Our job is to determine analytically just how that diversion takes place.

We consider the circular obstacle to be given by the equation $x^2+y^2 = 1$. Elementary symmetry considerations allow us to restrict attention to the flow in the upper half-plane. See Figure 14.9.

The boundary of the region W of the flow (Figure 14.9) is mapped to the boundary of the upper half plane U in the w variable (this boundary is just the u-axis) by the conformal mapping

$$w = f(z) = z + \frac{1}{z}. \tag{14.3.3.1.8}$$

And the region itself (shaded in Figure 14.10) is mapped to the upper half plane in the w variable. The complex potential for a uniform flow in the upper half plane of the w variable is

$$G(w) = Aw, \tag{14.3.3.1.9}$$

where A is a positive constant. Composing this potential with the mapping f, we find that the corresponding potential on W is

$$H(z) = G \circ f(z) = A \cdot \left(z + \frac{1}{z}\right). \tag{14.3.3.1.10}$$

The velocity [referring to equation (14.3.3.1.6)] is then

$$V = \overline{H'(z)} = A\left(1 - \frac{1}{\overline{z}^2}\right). \tag{14.3.3.1.11}$$

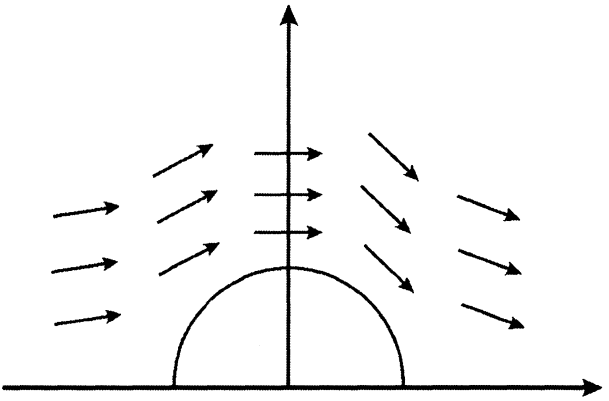


FIGURE 14.9
Restriction of attention to the flow in the upper half-plane.

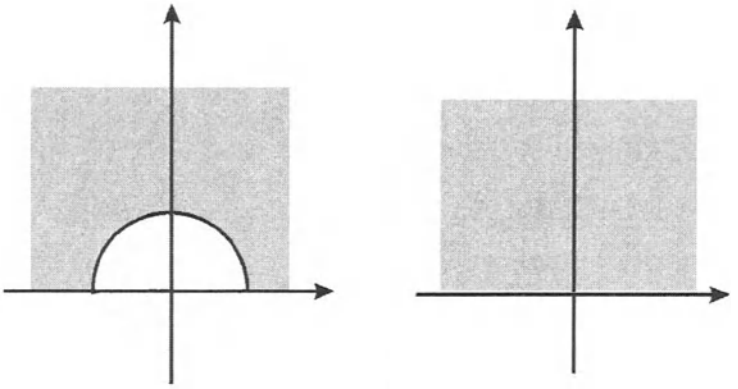


FIGURE 14.10
Conformal map of the region of the flow to the upper half-plane.

Note that V approaches A as $|z|$ increases monotonically to infinity. We conclude that the flow is almost uniform, and parallel to the x-axis, at points that are far from the circular obstacle. Finally observe that $V(\overline{z}) = \overline{V(z)}$, so that, by symmetry, formula (14.3.3.11) also represents the velocity of the flow in the lower half-plane.

According to equation (14.3.3.1.3), the stream function for our problem, written in polar coordinates, is just the imaginary part of H, or

$$\psi = A\left(r - \frac{1}{r}\right)\sin\theta. \qquad (14.3.3.1.12)$$

The streamlines

$$A\left(r - \frac{1}{r}\right)\sin\theta = C \qquad (14.3.3.1.13)$$

are symmetric to the y-axis and have asymptotes parallel to the x-axis. When $C = 0$, then the streamline consists of the circle $r = 1$ and the parts of the x axis that lie outside the unit circle. □

14.4 Numerical Techniques of Conformal Mapping

In practical applications, computer techniques for calculating conformal mappings have proved to be decisive. For instance, it is a standard technique to conformally map the complement of an airfoil (Figure 14.11) to the complement of a circle in order to study a boundary value problem on the complement. Of course the boundary of the airfoil is not given by any standard geometric curve (circle, parabola, ellipse, etc.), and the only hope of getting accurate information about the conformal mapping is by means of numerical analysis.

The literature on numerical techniques of conformal mapping is extensive. The reference [KYT] is a gateway to some of the standard references. Here we present only a brief sketch of some of the ideas.

14.4.1 Numerical Approximation of the Schwarz-Christoffel Mapping

Let P be a polygon in the complex plane with vertices $w_1, \ldots, w_n$. We wish to conformally map, by way of a mapping g, the upper half plane U to the interior of P—see Figure 14.12. The vertices $w_1, \ldots, w_{n-1}$ in the boundary P will have pre-images $x_1, \ldots, x_{n-1}$ under g in ∂U. These latter points are called *pre-vertices.* It is standard to take $\pm\infty \in \partial U$ to be the

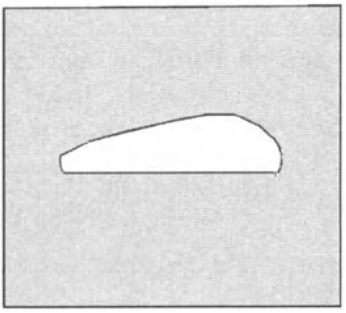


FIGURE 14.11
An airfoil.

pre-images of the last vertex w_n. Recall that, associated to each corner w_j, we have a "right-turn angle" θ_j. See Figure 14.13.

A conformal self-map of the unit disc is completely determined once the images of three boundary points are known. Since the disc and the upper half plane are conformally equivalent by way of the Cayley transform (§§6.3.5, 6.3.7, 12.2.2), it follows that, in specifying the Schwarz-Christoffel map, we may select three of the x_j's arbitrarily.

We will take $x_1 = -1, x_2 = 0$, and $x_n = +\infty$. It remains to determine the other $n-3$ vertices (which may *not* be freely chosen). The Schwarz-Christoffel map has the form

$$A \oint d\zeta + B \tag{14.4.1.1}$$

(see Figure 14.26). The choice of A will determine the size of the image and the choice of B will determine the position. Thus we need to choose $x_3, \ldots, x_{n-1}$ so that the image mapping has the right *shape*.

Specifying that the image of the Schwarz-Christoffel map g has the pre-specified shape (that is, the shape of P) is equivalent to demanding that

$$\frac{|g(x_j) - g(x_{j-1})|}{|g(x_2) - g(x_1)|} = \frac{|w_j - w_{j-1}|}{|w_2 - w_1|}, \qquad j = 3, 4, \ldots, n-1. \tag{14.4.1.2}$$

The theory of Schwarz-Christoffel guarantees that the angles in the image will be correct, and the $n-3$ equations (14.4.1.2) force $n-2$ of the side lengths to have the correct proportions. With a little planar geometry, one sees that this information in turn completely determines the shape of the image of the mapping g.

Finding the correct pre-vertices for a Schwarz-Christoffel mapping problem is called the *Schwarz-Christoffel parameter problem*. The problem is

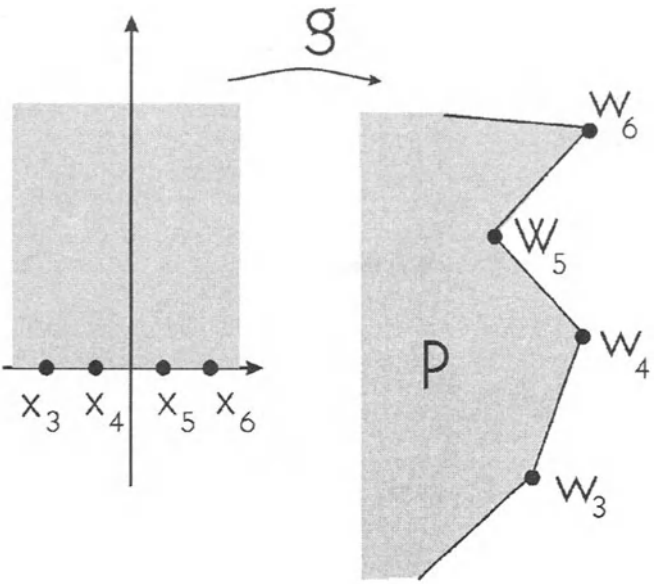


FIGURE 14.12
Mapping the upper half-plane to a polygonal region.

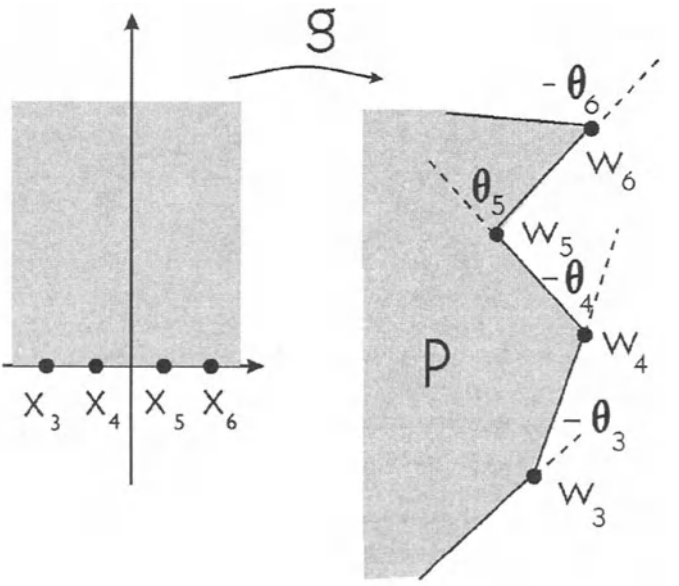


FIGURE 14.13
Vertices, pre-vertices, and right-turn angles.

resolved by solving the constrained system (14.4.1.2) of $n-3$ nonlinear equations. The constraint is that $0 < x_3 < x_4 < \cdots < x_{n-1} < \infty$. Unfortunately, standard numerical solution techniques (such as Newton's method and its variants) do not allow for constraints such as these. We can eliminate the constraint with a change of variable: Set

$$\widetilde{x}_j = \log(x_j - x_{j-1}) \,, \qquad j = 3, 4, \ldots, n-1. \tag{14.4.1.3}$$

The $\tilde{x}_j$'s are arbitrary real numbers, with no constraints. The new system, expressed in terms of the variables $\widetilde{x}_3, \widetilde{x}_4, \ldots, \widetilde{x}_{n-1}$, can be solved using a suitable version of Newton's method. The necessary algorithm is available as part of MATLAB, Mathematica, and also on most large scientific computer installations.

Once we have solved the Schwarz-Christoffel parameter problem, then we wish to calculate the actual conformal map. This entails calculating the Schwarz-Christoffel integral

$$\oint_0^z (\zeta - x_1)^{\theta_1/\pi} (\zeta - x_2)^{\theta_2/\pi} \cdots (\zeta - x_{n-1})^{\theta_{n-1}/\pi} \, d\zeta. \tag{14.4.1.4}$$

Even when P is a very simple polygon such as a triangle, we cannot expect to evaluate the integral (14.4.1.4) by hand. There are well-known numerical techniques for evaluating an integral $\int_a^b \varphi(t)\, dt$. Let us describe three of them: Fix a partition $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$ of the interval of integration, which is such that each interval in the partition has the same length. Let $\Delta t = (b-a)/k$ denote that common length.

I. The Midpoint Rule Set $p_j = a + (j - 1/2)\Delta t$, $j = 1, \ldots, k$. Then

$$\int_a^b \varphi(t)\, dt \approx \sum_{j=1}^{k} \varphi(p_j)\Delta t. \tag{14.4.1.5}$$

If φ is smooth, then the error in this calculation is of size k^{-2}.

II. The Trapezoid Rule In this methodology we take the points p_j at which φ is evaluated to be the interval endpoints. We have

$$\int_a^b \varphi(t)\, dt \approx \frac{\Delta t}{2} \left[\varphi(t_0) + 2\varphi(t_1) + 2\varphi(t_2) + \cdots + 2\varphi(t_{k-1}) + \varphi(t_k)\right]. \tag{14.4.1.6}$$

If φ is smooth, then the error in this calculation is of size k^{-3}.

III. Simpson's Rule In this methodology we use both the interval endpoints and the midpoints. We have

$$\int_a^b \varphi(t)\,dt \approx \frac{\Delta t}{6}\Big\{\varphi(t_0)+\varphi(t_k)+2\big[\varphi(t_1)+\cdots\varphi(t_{k-1})\big] +4\left[\varphi\Big(\frac{t_0+t_1}{2}\Big)+\cdots+\varphi\Big(\frac{t_{k-1}+t_k}{2}\Big)\right]\Big\}. \tag{14.4.3.7}$$

If φ is smooth, then the error in this calculation is of size k^{-4}.

More sophisticated techniques, such as the *Newton-Cotes formulas* and *Gaussian quadrature* give even more accurate approximations.

Unfortunately, the integrand in the Schwarz-Christoffel integral has singularities at the pre-vertices x_j. Thus we must use a variant of the above-described numerical integration techniques with the partition points and with weights chosen so as to compensate for the singularities. These ideas are encapsulated in the method of *Gauss-Jacobi quadrature.*

Finally, it is often the case that the pre-vertices are very close together. This extreme proximity can work against the compensating properties of Gauss-Jacobi quadrature. The method of *compound Gauss-Jacobi quadrature* mandates that difficult intervals be heavily subdivided near the endpoints. This method results in a successful calculation of the Schwarz-Christoffel mapping.

14.4.2 Numerical Approximation to a Mapping onto a Smooth Domain

One can construct a numerical approximation to a conformal mapping of the upper half-plane U onto a smoothly bounded domain Ω with boundary curve C by thinking of C as a polygon with infinitely many corners w_j, each with infinitesimal turning angle θ_j.

From the Schwarz-Christoffel formula for the mapping g that we discussed in §§14.4.1, we know that

$$\begin{aligned} g'(z) &= A(z-x_1)^{\theta_1/\pi}(z-x_2)^{\theta_2/\pi}(z-x_{n-1})^{\theta_{n-1}/\pi} \\ &= A\exp\left[\frac{1}{\pi}\sum_{j=1}^{n-1}\theta_j\log(z-x_j)\right]. \end{aligned} \tag{14.4.2.1}$$

As $n \to +\infty$, it is natural to think of the sum as converging to an integral. So we have derived the formula

$$g'(z) = A\exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\theta(x)\log(z-x)\,dx\right]. \tag{14.4.2.2}$$

Here θ is a function that describes the turning of the curve C per unit length along the x-axis. Integrating, we find that

$$g(z) = A \int_0^z \exp\left[\frac{1}{\pi} \int_{-\infty}^{\infty} \theta(x) \log(z - x)\, dx\right] + B. \qquad (14.4.2.3)$$

Of course the discussion here has only been a sketch of some of the key ideas associated with numerical conformal mapping. The reference [SASN, pp. 430-443] gives a more discursive discussion, with detailed examples. The reference [SASN] also offers a further guide to the literature.

Appendix to Chapter 14

A Pictorial Catalog of Conformal Maps

Here we present a graphical compendium of commonly used conformal mappings. Most of the mappings that we present here are given by explicit formulas, and are also represented in figures. Wherever possible, we also provide the inverse of the mapping.

In each case, the domain of the mapping is the z-plane, with $x + iy = z \in \mathbb{C}$. And the range of the mapping is the w-plane, with $u + iv = w \in \mathbb{C}$. In some examples, it is appropriate to label special points $a, b, c, \ldots$ in the domain of the mapping and then to specify their image points $A, B, C, \ldots$ in the range. In other words, if the mapping is called f, then $f(a) = A$, $f(b) = B$, $f(c) = C$, etc.

All of the mappings presented here map the shaded region in the z-plane *onto* the shaded region in the w-plane. Most of the mappings are one-to-one (i.e., they do *not* map two distinct points in the domain of the mapping to the same point in the range of the mapping). In a few cases the mapping is *not* one-to-one; these few examples will be clear from context.

In the majority of these examples, the mapping is given by an explicit formula. In a few cases, such as the Schwarz-Christoffel mapping, the mapping is given by a semi-explicit integral. In one example (Figure 14.25), the mapping is given by an elliptic integral. Such integrals cannot be evaluated in closed form. But they can be calculated to any degree of accuracy using methods of numerical integration. Section 14.4 will also provide some information about numerical techniques of conformal mapping. The book [KOB] gives an extensive listing of explicit conformal mappings; see also [CCP]. The book [NEH] is a classic treatise on the theory of conformal mappings.

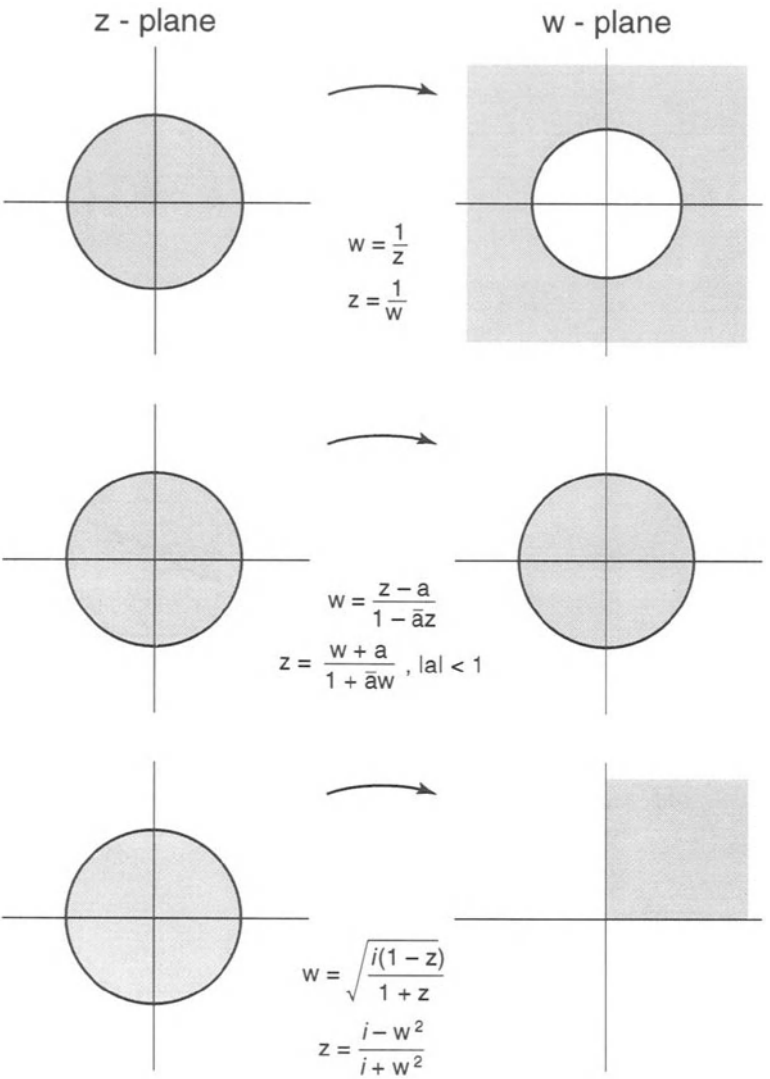


FIGURE 14.14
(top) *Map of the disc to its complement;* **(middle)** *Map of the disc to the disc;* **(bottom)** *Map of the upper half-disc to the first quadrant.*

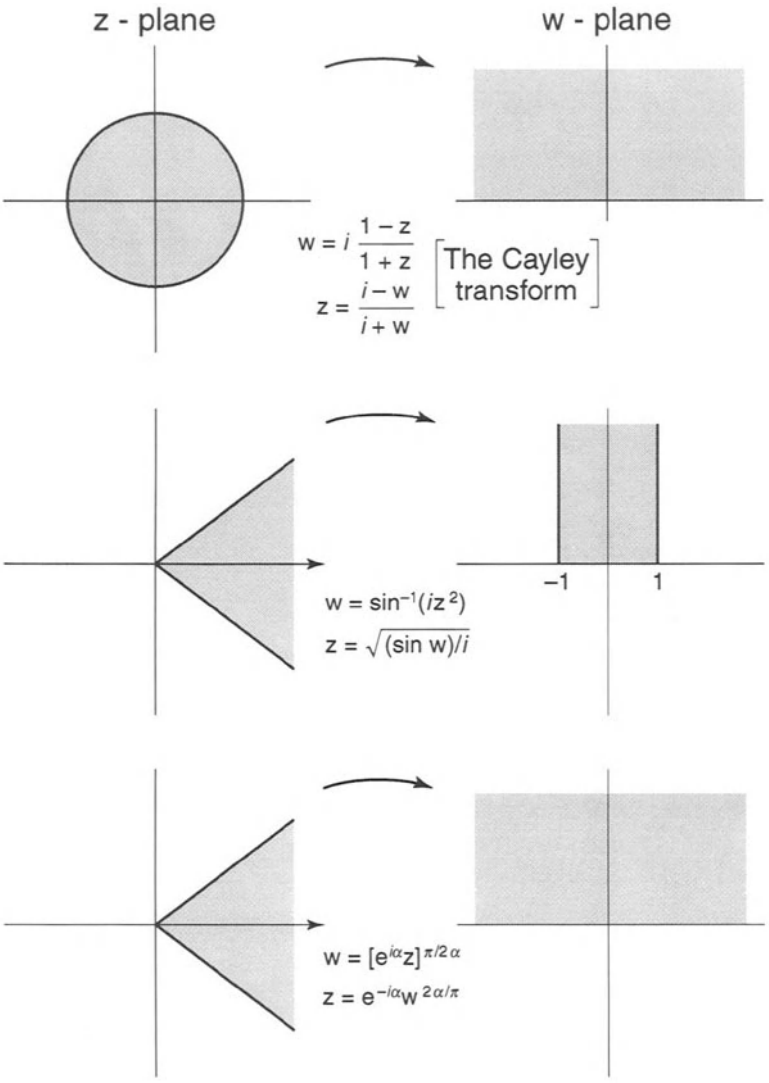


FIGURE 14.15
(top) *The Cayley transform: A map of the disc to the upper half-plane;* **(middle)** *Map of a cone to a half-strip;* **(bottom)** *Map of a cone to the upper half-plane.*

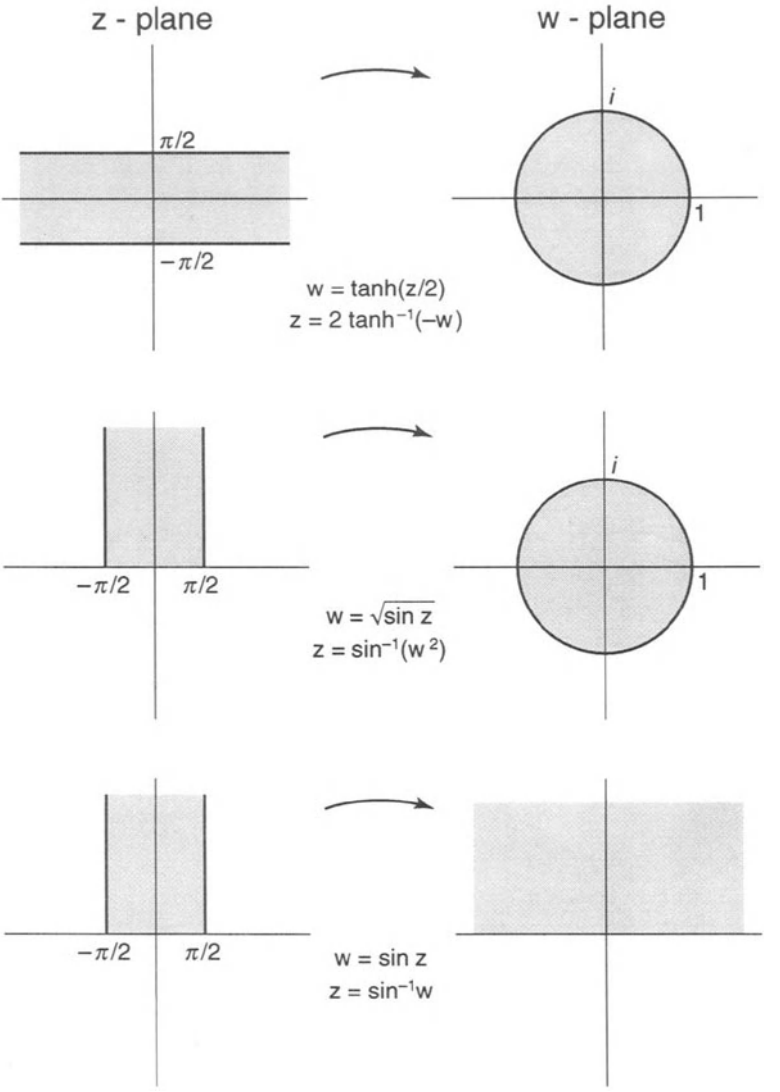


FIGURE 14.16
(top) *Map of a strip to the disc;* **(middle)** *Map of a half-strip to the disc;* **(bottom)** *Map of a half-strip to the upper half-plane.*

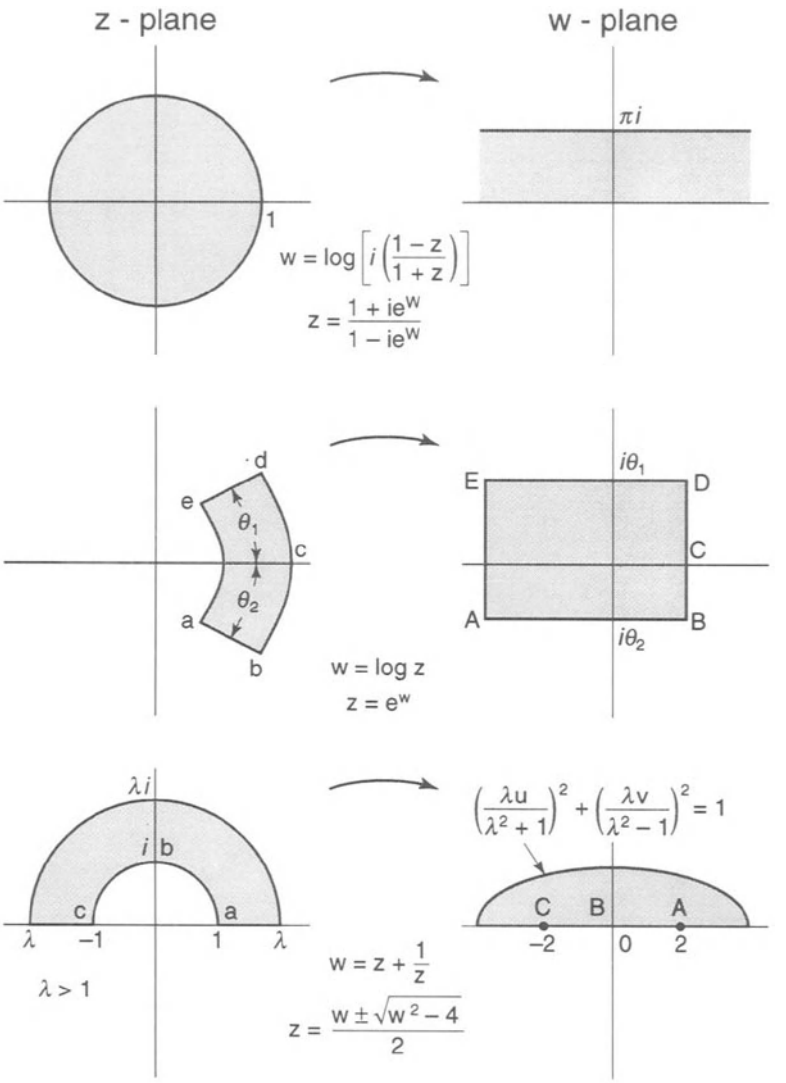


FIGURE 14.17
(top) *Map of the disc to a strip;* **(middle)** *Map of an annular sector to the interior of a rectangle;* **(bottom)** *Map of a half-annulus to the interior of a half-ellipse.*

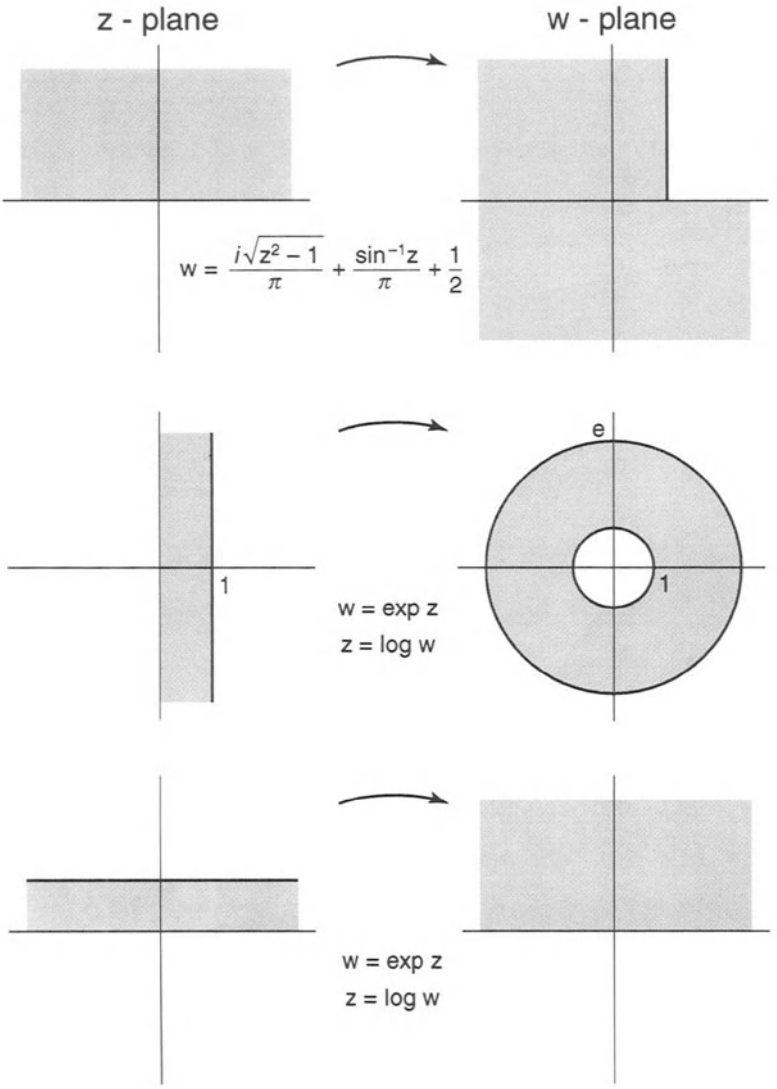


FIGURE 14.18
(top) *Map of the upper half-plane to a* 3/4*-plane;* **(middle)** *Map of a strip to an annulus;* **(bottom)** *Map of a strip to the upper half-plane.*

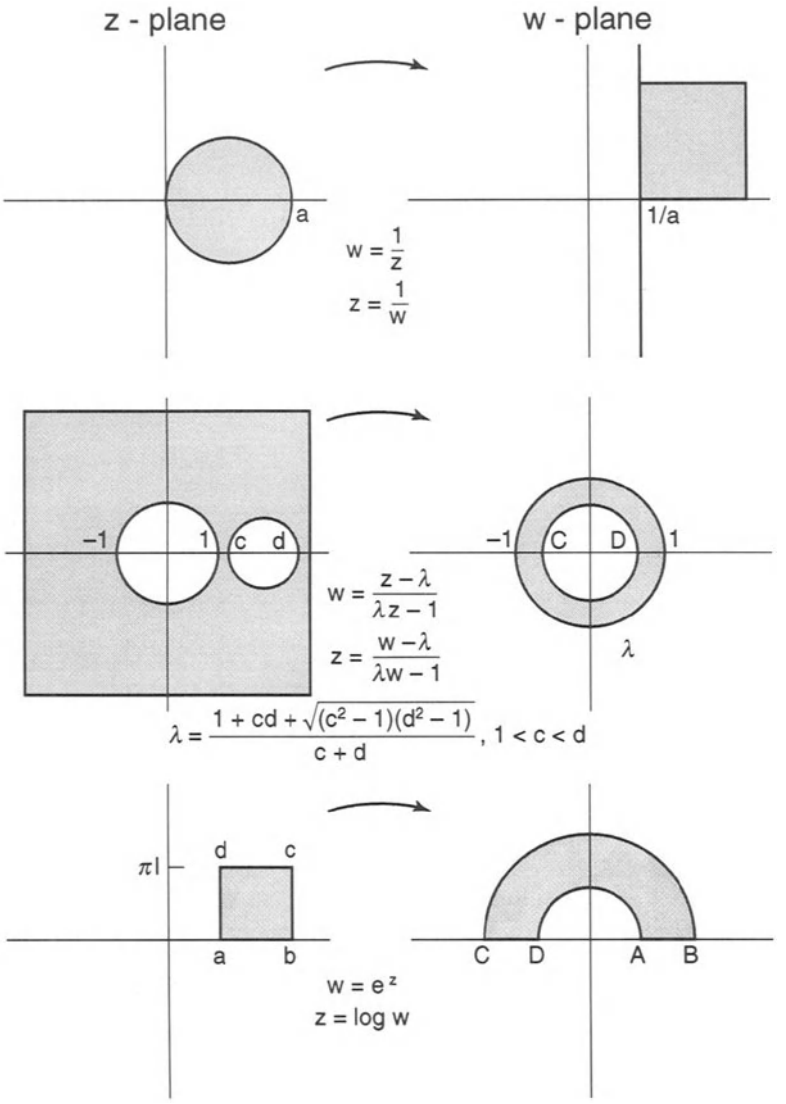


FIGURE 14.19
(top) *Map of a disc to a quadrant;* **(middle)** *Map of the complement of two discs to an annulus;* **(bottom)** *Map of the interior of a rectangle to a half-annulus.*

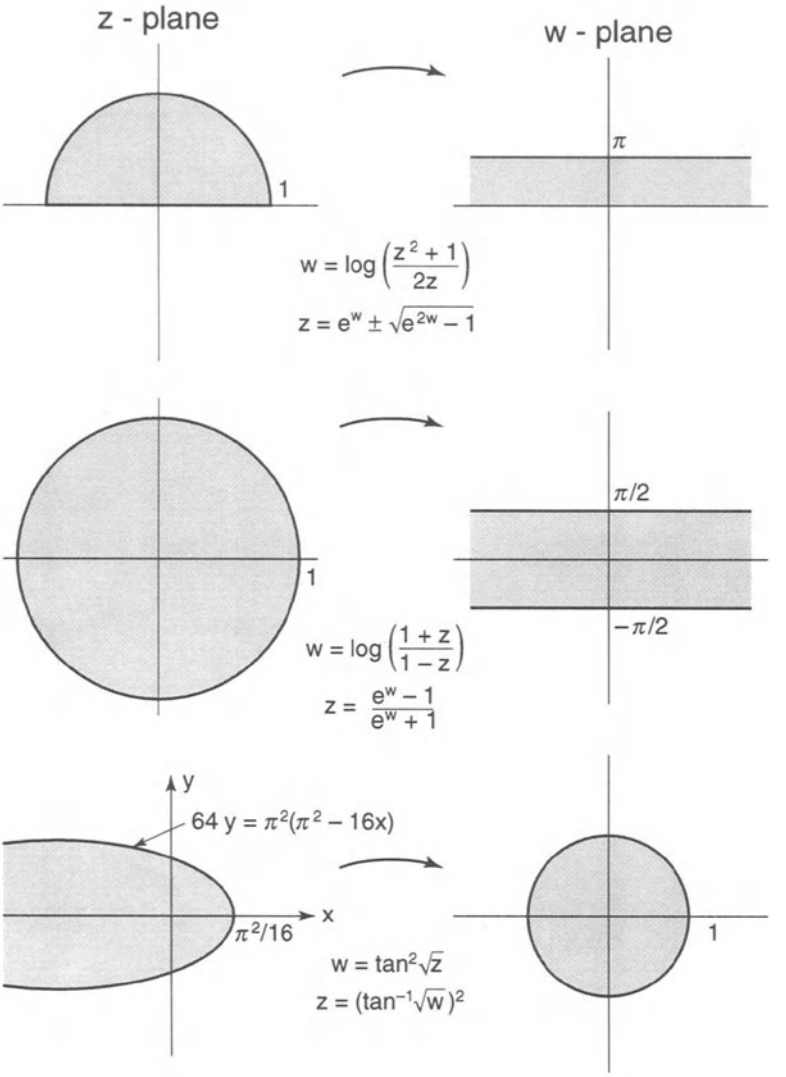


FIGURE 14.20
(top) *Map of a half-disc to a strip;* **(middle)** *Map of a disc to a strip;* **(bottom)** *Map of the inside of a parabola to a disc.*

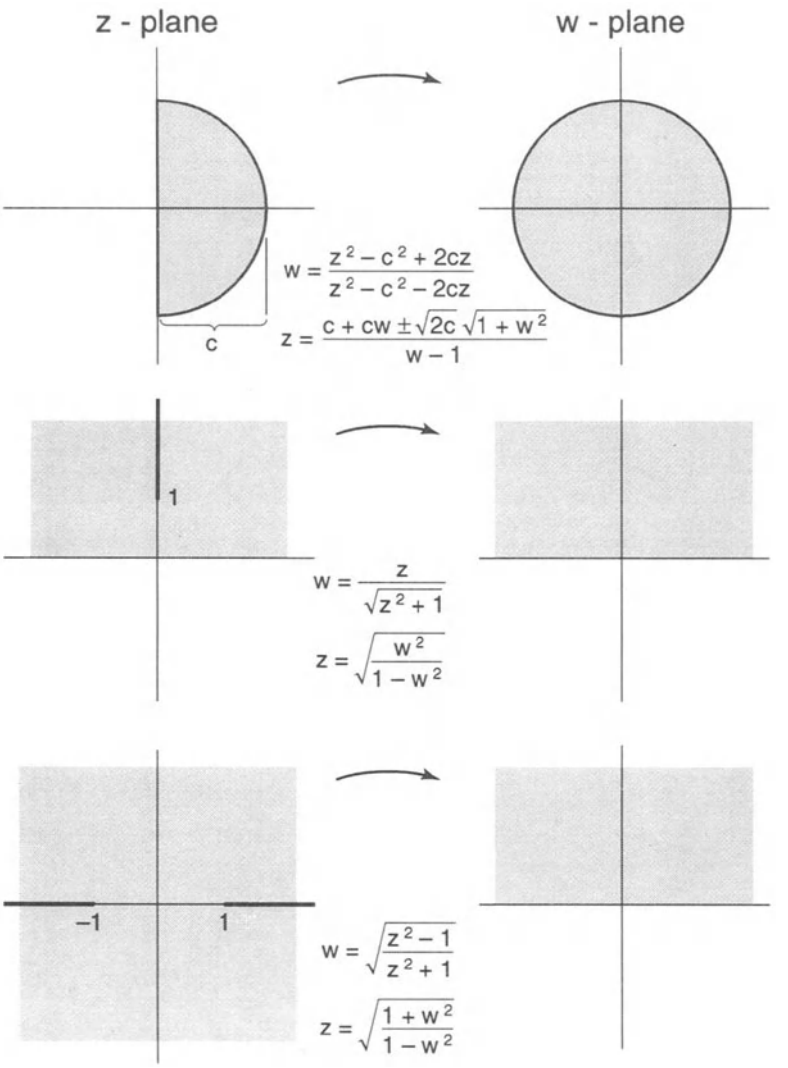

z - plane
w - plane
c
$w = \frac{z^2 - c^2 + 2cz}{z^2 - c^2 - 2cz}$
$z = \frac{c + cw \pm \sqrt{2c}\sqrt{1 + w^2}}{w - 1}$
1
$w = \frac{z}{\sqrt{z^2 + 1}}$
$z = \sqrt{\frac{w^2}{1 - w^2}}$
−1
1
$w = \sqrt{\frac{z^2 - 1}{z^2 + 1}}$
$z = \sqrt{\frac{1 + w^2}{1 - w^2}}$

FIGURE 14.21
(top) *Map of a half-disc to a disc;* **(middle)** *Map of the slotted upper half-plane to upper half-plane;* **(bottom)** *Map of the double-sliced plane to the upper half-plane.*

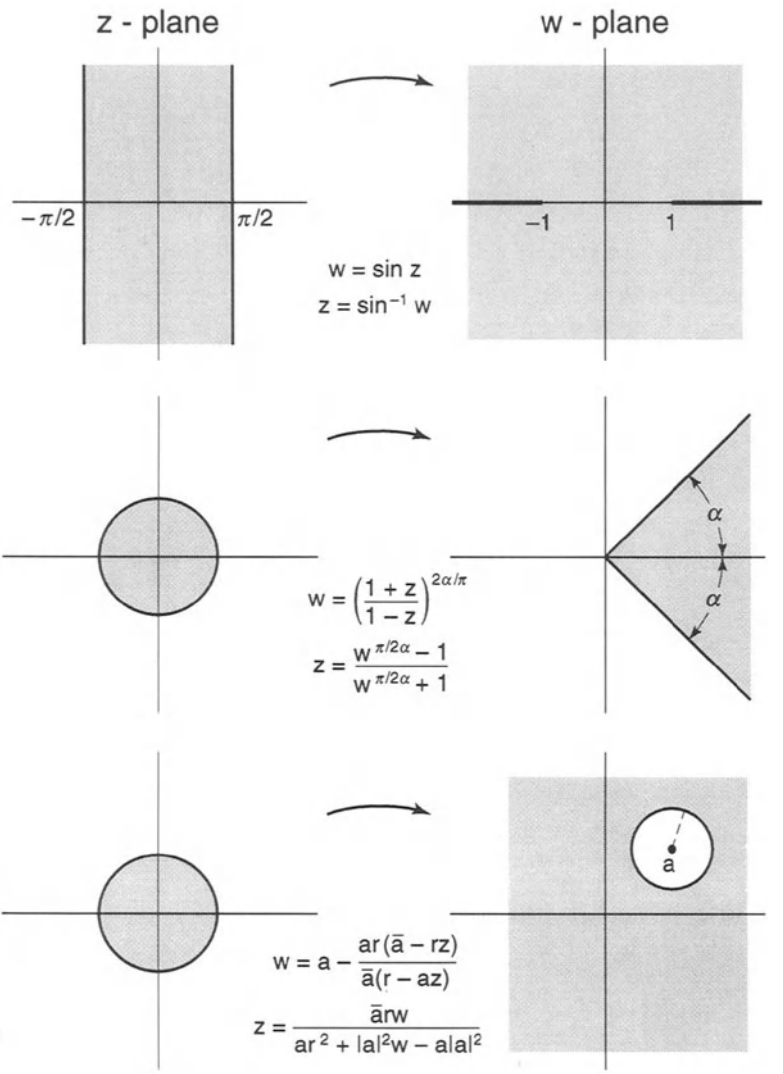


FIGURE 14.22
(top) *Map of a strip to the double-sliced plane;* **(middle)** *Map of the disc to a cone;* **(bottom)** *Map of a disc to the complement of a disc.*

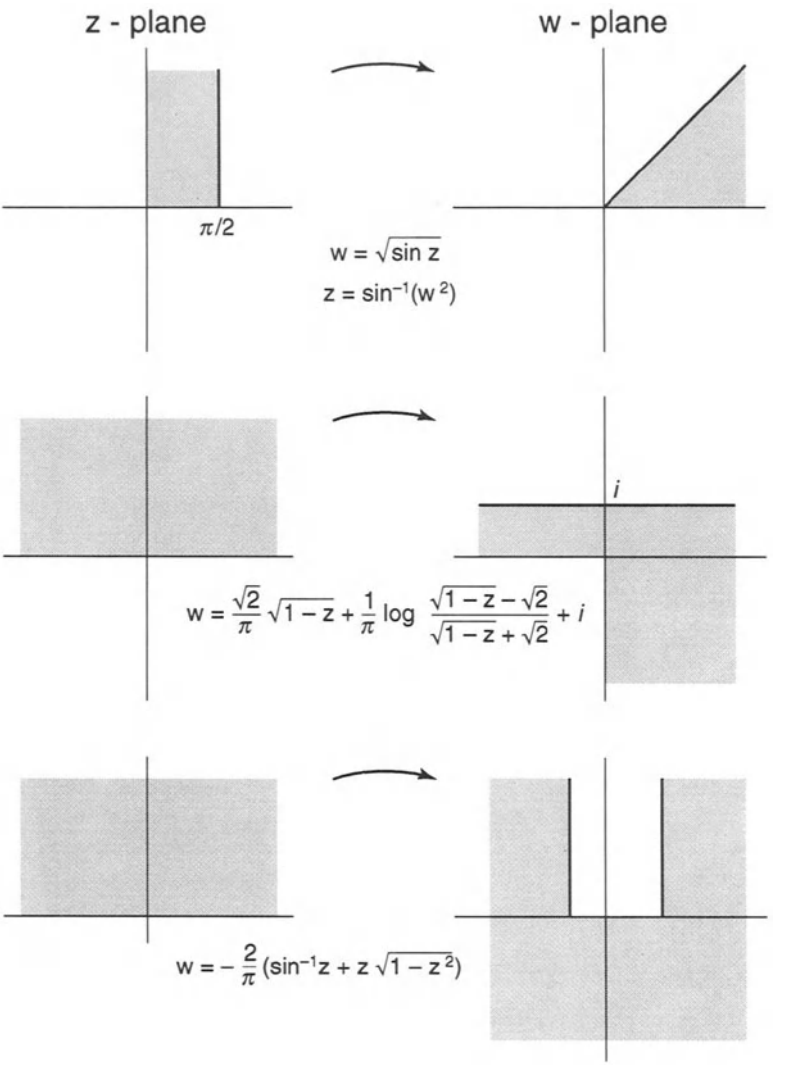


FIGURE 14.23
(top) *Map of a half-strip to a half-quadrant;* **(middle)** *Map of the upper half-plane to a right-angle region;* **(bottom)** *Map of the upper half-plane to the plane less a half-strip.*

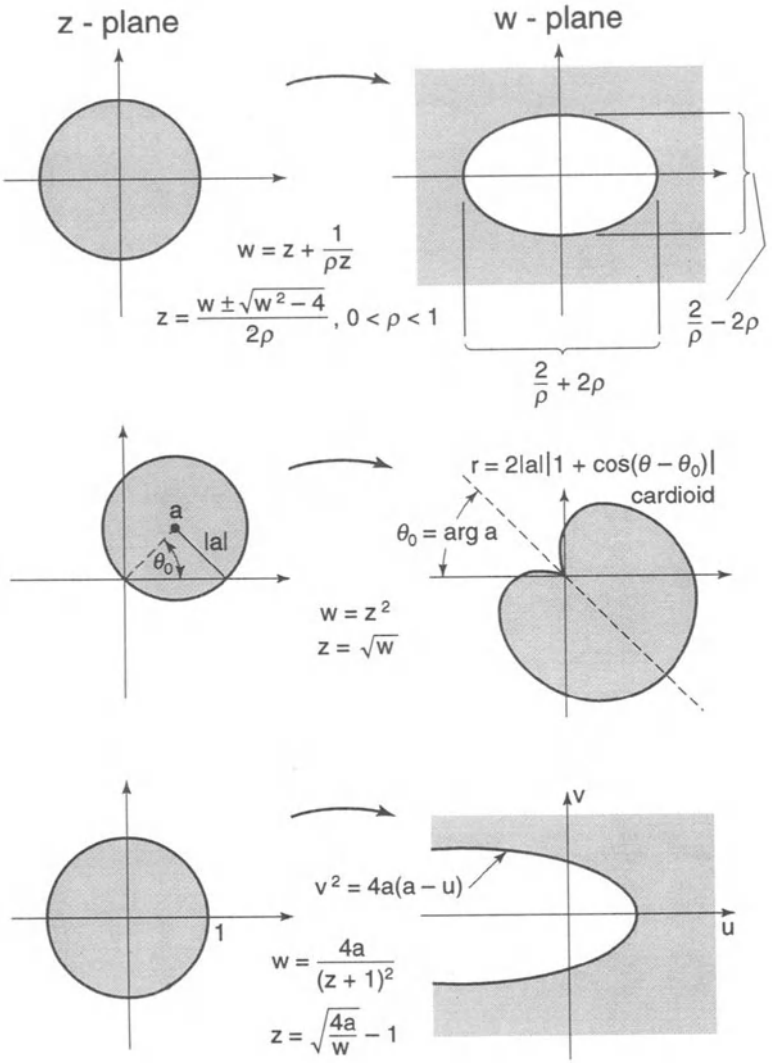


FIGURE 14.24
(top) *Map of a disc to the complement of an ellipse;* **(middle)** *Map of a disc to the interior of a cardioid;* **(bottom)** *Map of a disc to the region outside a parabola.*

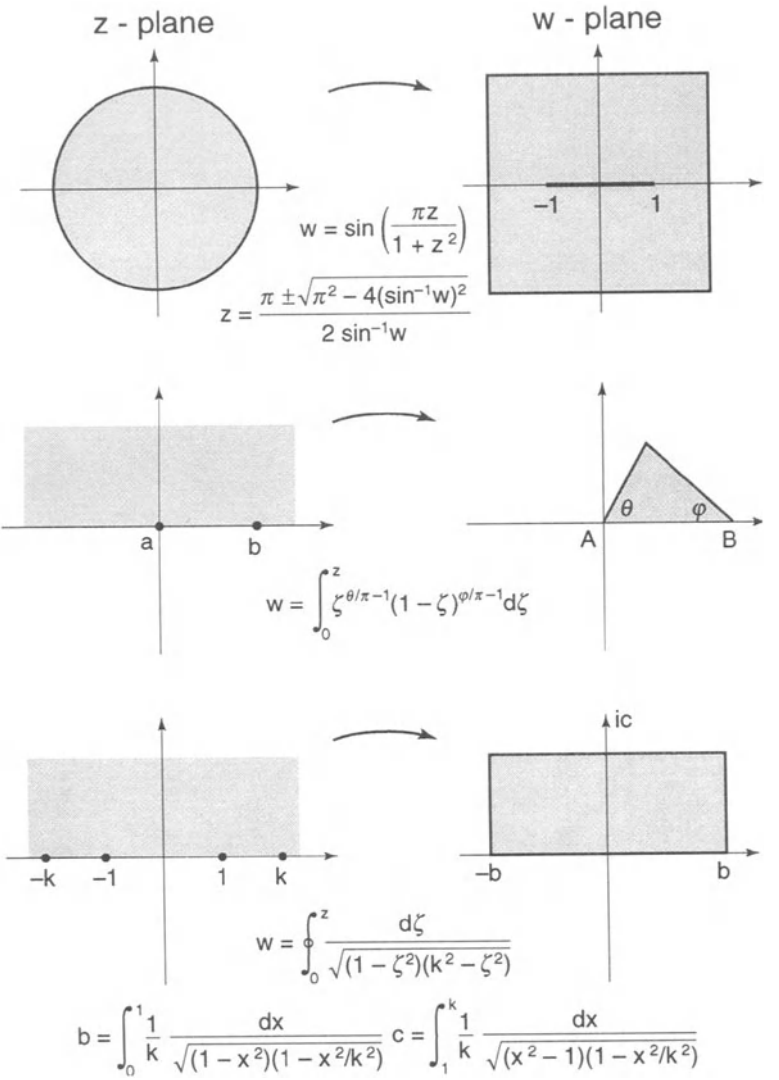
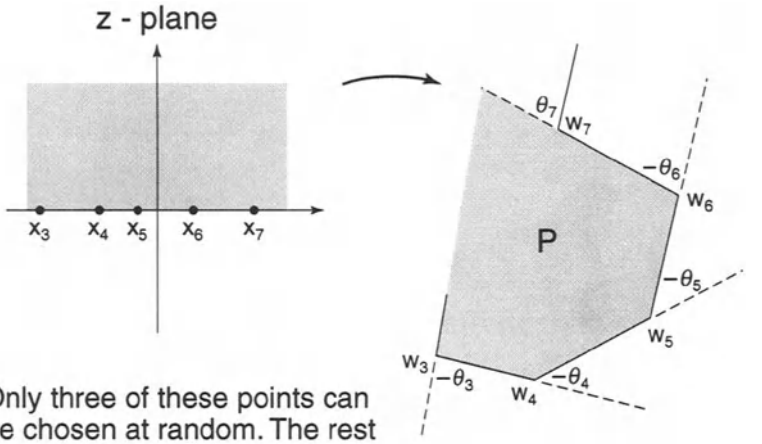


FIGURE 14.25
(top) *Map of the disc to the slotted plane;* **(middle)** *Map of the upper half-plane to the interior of a triangle;* **(bottom)** *Map of the upper half-plane to the interior of a rectangle.*



$$w = f(z) = A\int_0^z (\zeta - x_1)^{\theta_1/\pi}\cdot(\zeta - x_2)^{\theta_2/\pi}\cdots(\zeta - x_{n-1})^{\theta_{n-1}/\pi}d\zeta + B$$

$$f(x_1) = w_1,\ f(x_2) = w_2,\ \cdots,\ f(x_n) = w_n$$

FIGURE 14.26
The Schwarz-Christoffel formula.

Chapter 15

Transform Theory

15.0 Introductory Remarks

This chapter will sketch some connections of Fourier series, the Fourier transform, and the Laplace transform with the theory of complex variables. This will not be a tutorial in any of these three techniques. The reader who desires background should consult the delightful texts [DYM] or [KAT].

The idea of Fourier series or Fourier transforms is to take a function f that one wishes to analyze and to assign to f a new function $\widehat{f}$ that contains information about the frequencies that are built into the function f. As such, the Fourier theory is a real variable theory. But complex variables can come to our aid in the calculation of, and also in the analysis of, $\widehat{f}$. It is that circle of ideas that will be explained in the present chapter.

15.1 Fourier Series

15.1.1 Basic Definitions

Fourier series takes place on the interval $[0, 2\pi)$. We think of the endpoints of this interval as being identified, so that geometrically our analysis is taking place on a circle (Figure 15.1). If f is an integrable function on $[0, 2\pi)$, then we *define*

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt. \tag{15.1.1.1}$$

The *Fourier series* of f is the formal expression

$$Sf(t) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{int}. \tag{15.1.1.2}$$

S. G. Krantz, *Handbook of Complex Variables*
© Springer Science+Business Media New York 1999

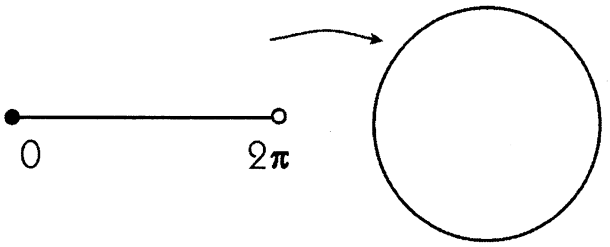


FIGURE 15.1
Identification of the interval with the circle.

We call this a "formal expression" because we do not know a priori whether this series converges in any sense, and if it does converge, whether its limit is the original function f.

There is a highly developed theory of the convergence of Fourier series, but this is not the proper context in which to describe those results. Let us simply formulate one of the most transparent and useful theorems.

The partial sums of the Fourier series Sf are defined to be

$$S_N f(t) \equiv \sum_{n=-N}^{N} \widehat{f}(n) e^{int}. \tag{15.1.1.3}$$

We say that the Fourier series *converges* to the function f at the point t if

$$\lim_{N \to \infty} S_N f(t) = f(t). \tag{15.1.1.4}$$

> **Theorem:** Let f be an integrable function on $[0, 2\pi)$. If t_0 is a point of differentiability of f, then the Fourier series Sf converges to f at t_0.

Many functions that we encounter in practice are piecewise differentiable, so this is a theorem that is straightforward to apply. In fact, if f is continuously differentiable on a compact interval I, then the Fourier series converges absolutely and uniformly to the original function f. In this respect Fourier series are much more attractive than Taylor series; for the Taylor series of even a C^∞ function f typically does not converge, and even when it does converge, it typically does not converge to f.

15.1.2 A Remark on Intervals of Arbitrary Length

It is sometimes convenient to let the interval $[-\pi, \pi]$ be the setting for our study of Fourier series. Since we think of the function f as being 2π-periodic, this results in no change in the notation or in the theory.

In applications, one often wants to do Fourier series analysis on an interval $[-L/2, L/2]$. In this setting the notation is adjusted as follows: For a function f that is integrable on $[-L/2, L/2]$, we define

$$\widehat{f}(n) = \frac{1}{L}\int_{-L/2}^{L/2} f(t)e^{-in2\pi/L}\,dt \tag{15.1.2.1}$$

and set the Fourier series of f equal to

$$Sf(t) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in2\pi t/L}. \tag{15.1.2.2}$$

We shall say no more about Fourier analysis on $[-L/2, L/2]$ at this time.

15.1.3 Calculating Fourier Coefficients

The key to using complex analysis for the purpose of computing Fourier series is to note that, when $n \geq 0$, the function $\varphi_n(t) = e^{int}$ is the "boundary function" of the holomorphic function z^n. What does this mean?

We identify the interval $[0, 2\pi)$ with the unit circle S in the complex plane by way of the map

$$\begin{aligned} M : [0, 2\pi) &\longrightarrow S \\ t &\longmapsto e^{int}. \end{aligned} \tag{15.1.3.1}$$

Of course, the circle S is the boundary of the unit disc D. If we let z be a complex variable, then, when $|z| = 1$, we know that z has the form $z = e^{it}$ for some real number t between 0 and 2π. Thus the holomorphic (analytic) function z^n, $n \geq 0$, takes the value $(e^{it})^n = e^{int}$ on the circle. By the same token, when $n < 0$, then the meromorphic function z^n takes the value $(e^{it})^n = e^{int}$ on the circle. In this way we associate, in a formal fashion, the meromorphic function

$$F(z) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)z^n \tag{15.1.3.2}$$

with the Fourier series

$$Sf(t) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{int}. \tag{15.1.3.3}$$

This association is computationally useful, as the next example shows.

15.1.4 Calculating Fourier Coefficients Using Complex Analysis

Let us calculate the Fourier coefficients of the function $f(t) = e^{2i\sin t}$ using complex variable theory. We first recall that

$$\sin t = \frac{1}{2i}\left[e^{it} - e^{-it}\right]. \tag{15.1.4.1}$$

Thus, using the ideas from §§15.1.3, we associate to $2i\sin t$ the analytic function

$$z - \frac{1}{z}. \tag{15.1.4.2}$$

As a result, we associate to f the analytic function

$$F(z) = e^{z-1/z} = e^z \cdot e^{-z^{-1}}. \tag{15.1.4.3}$$

But the function on the right is easy to expand in a series:

$$\begin{aligned} F(z) &= e^z \cdot e^{-z^{-1}} \\ &= \left[\sum_{k=0}^{\infty} \frac{z^k}{k!}\right] \cdot \left[\sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{-\ell}}{\ell!}\right]. \end{aligned} \tag{15.1.4.4}$$

By the theory of the Cauchy product of series (see [KRA2]), two convergent power series may be multiplied together in just the same way as two polynomials: we multiply term by term and then gather together the resulting terms with the same power of z. We therefore find that

$$F(z) = \sum_{n=-\infty}^{\infty} z^n \left[\sum_{m=n}^{\infty} \frac{1}{m!}\frac{(-1)^n}{(m-n)!}\right]. \tag{15.1.4.5}$$

In conclusion, we see that the Fourier series of our original function f is

$$Sf(x) \sim \sum_{-\infty}^{\infty} \widehat{f}(n) e^{inx} \tag{15.1.4.6}$$

with

$$\widehat{f}(n) = \sum_{m=n}^{\infty} \frac{1}{m!}\frac{(-1)^n}{(m-n)!}. \tag{15.1.4.7}$$

□

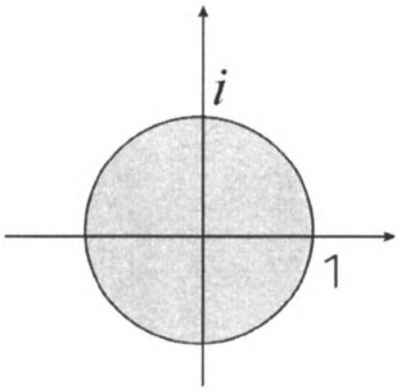


FIGURE 15.2
Mathematical model of the disc.

15.1.5 Steady State Heat Distribution

The next example will harken back to our discussion of heat diffusion in §§14.2.2. But we will now use some ideas from Fourier series and from Laurent series.

EXAMPLE 15.1.5.1 Suppose that a thin metal heat-conducting plate is in the shape of a round disc and has radius 1. Imagine that the upper half of the circular boundary of the plate is held at constant temperature 10° and the lower half of the circular boundary is held at constant temperature 0°. Describe the steady state heat distribution on the entire plate.

Solution: Model the disc with the interior of the unit circle in the complex plane (Figure 15.2). Identifying $[0, 2\pi)$ with the unit circle as in §§15.1.3, we are led to consider the function

$$f(t) = \begin{cases} 10 \text{ if } 0 \leq t \leq \pi \\ 0 \quad \text{if } \pi \leq t < 2\pi. \end{cases} \tag{15.1.5.1.1}$$

Then

$$\widehat{f}(0) = \frac{1}{2\pi} \int_0^{\pi} 10\, dt + \frac{1}{2\pi} \int_{\pi}^{2\pi} 0\, dt = 5 \tag{15.1.5.1.2}$$

and, for $n \neq 0$,

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{\pi} 10 \cdot e^{-int}\, dt + \frac{1}{2\pi} \int_{\pi}^{2\pi} 0 \cdot e^{-int}\, dt = \frac{1}{2\pi} \frac{10}{in} \left[1 - e^{-in\pi}\right]. \tag{15.1.5.1.3}$$

As a result,

$$\begin{aligned} Sf &\sim 5 + \sum_{-\infty}^{-1} \frac{1}{2\pi} \cdot \frac{10}{in} \left[1 - e^{-in\pi}\right] e^{int} + \sum_{n=1}^{\infty} \frac{1}{2\pi} \cdot \frac{10}{in} \left[1 - e^{-in\pi}\right] e^{int} \\ &= 5 + \sum_{n=1}^{\infty} \frac{1}{2\pi} \cdot \frac{10}{-in} \left[1 - e^{in\pi}\right] e^{-int} + \sum_{n=1}^{\infty} \frac{1}{2\pi} \cdot \frac{10}{in} \left[1 - e^{-in\pi}\right] e^{int} \\ &= 5 + 2\mathrm{Re} \left(\sum_{n=1}^{\infty} \frac{1}{2\pi} \cdot \frac{10}{in} \left[1 - e^{-in\pi}\right] \right) e^{int} \end{aligned} \tag{15.1.5.1.4}$$

Of course the expression in brackets is 0 when n is even. So we can rewrite our formula for the Fourier series as

$$Sf \sim 5 + \frac{20}{\pi} \mathrm{Re} \left(\frac{1}{i} \sum_{k=0}^{\infty} \frac{1}{2k+1} e^{i(2k+1)t} \right). \tag{15.1.5.1.5}$$

This series is associated, just as we discussed in §§15.1.3, with the analytic function

$$F(z) = 5 + \frac{20}{\pi} \mathrm{Re} \left(\frac{1}{i} \sum_{k=0}^{\infty} \frac{1}{2k+1} z^{i(2k+1)t} \right). \tag{15.1.5.1.6}$$

To sum the series

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} z^{2k+1}, \tag{15.1.5.1.7}$$

we write it as

$$\sum_{k=0}^{\infty} \int_0^z \zeta^{2k}\, d\zeta = \int_0^z \left[\sum_{k=0}^{\infty} \zeta^{2k} \right] d\zeta, \tag{15.1.5.1.8}$$

where we have used the fact that integrals and convergent power series commute. But

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots \tag{15.1.5.1.9}$$

is a familiar power series expansion. Using this series with $\alpha = z^2$ we find that

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} z^{2k+1} = \int_0^z \frac{1}{1-\zeta^2}\, d\zeta = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right). \tag{15.1.5.1.10}$$

Putting this information into (15.1.5.1.6) yields

$$\begin{aligned} F(z) &= 5 + \frac{10}{\pi} \mathrm{Re} \left[\frac{1}{i} \log \left(\frac{1+z}{1-z} \right) \right] \\ &= 5 + \frac{10}{\pi} \arg \left(\frac{1+z}{1-z} \right). \end{aligned} \tag{15.1.5.1.11}$$

This function $F(z) = F(re^{i\theta})$ is the harmonic function on the disc with boundary function f. It is therefore the solution to our heat diffusion problem. □

15.1.6 The Derivative and Fourier Series

Now we show how complex variables can be used to discover important formulas about Fourier coefficients. In this subsection we concentrate on the derivative.

Let f be a C^1 function on $[0, 2\pi)$. Assume that $f(0) = \lim_{t \to 2\pi^-} f(t)$ and that $f'(0) = \lim_{t \to 2\pi^-} f'(t)$, so that the values of f and its derivative match up at the endpoints. We want to consider how the Fourier series of f relates to the Fourier series of f'. We proceed formally.

We write

$$f(t) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{int} \tag{15.1.6.1}$$

and hence

$$F(z) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) z^n. \tag{15.1.6.2}$$

A convergent power series may be differentiated termwise, so we have

$$F'(z) = \frac{dF}{dz}(z) = \sum_{n=-\infty}^{\infty} n\widehat{f}(n) z^{n-1}. \tag{15.1.6.3}$$

But, with $z = e^{it}$, we have

$$\frac{dz}{dt} = \frac{d}{dt} e^{it} = ie^{it} = iz,$$

so the chain rule tells us that

$$\begin{aligned} \frac{dF}{dt} &= \frac{dF}{dz} \cdot \frac{dz}{dt} \\ &= \sum_{n=-\infty}^{\infty} n\widehat{f}(n) z^{n-1} \cdot (iz) \\ &= \sum_{n=-\infty}^{\infty} in\widehat{f}(n) z^n \\ &= \sum_{n=-\infty}^{\infty} in\widehat{f}(n) e^{int}. \end{aligned}$$

We conclude that the Fourier series for $f'(t)$ is

$$\sum_{n=-\infty}^{\infty} in\widehat{f}(n)e^{int}, \tag{15.1.6.4}$$

and that the Fourier coefficients for $f'(t)$ are

$$[f']\widehat{\ } = in\widehat{f}(n). \tag{15.1.6.5}$$

□

15.2 The Fourier Transform

The Fourier transform is the analogue on the real line of Fourier series coefficients for a function on $[0, 2\pi)$. For deep reasons (which are explained in [FOL]), the Fourier series on the bounded interval $[0, 2\pi)$ must be replaced by the continuous analogue of a sum, which is an integral. In this section we will learn what the Fourier transform is, and what the basic convergence question about the Fourier transform is. Then we will see how complex variable techniques may be used in the study of the Fourier transform.

15.2.1 Basic Definitions

The Fourier transform takes place on the real line $\mathbb{R}$. If f is an integrable function on $\mathbb{R}$, then we *define*

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi it\cdot\xi}\, dt. \tag{15.2.1.1}$$

The variable t is called the "space variable" and the variable ξ is called the "Fourier transform variable" (or sometimes the "phase variable"). There are many variants of this definition. Some tracts replace $-2\pi it \cdot \xi$ in the exponential with $2\pi it \cdot \xi$. Others omit the factor of 2π. We have chosen this particular definition because it simplifies certain basic formulas in the subject. The reader who wants to learn the full story of the theory of the Fourier transform should consult [FOL] or [STW].

The Fourier transform $\widehat{f}$ of an integrable function f enjoys the property that $\widehat{f}$ is continuous and vanishes at infinity. However $\widehat{f}$ *need not* be integrable. In fact $\widehat{f}$ can die arbitrarily slowly at infinity. This fact of life

necessitates extra care in formulating results about the Fourier transform and its inverse.

Recall that we recover a function on $[0, 2\pi)$ from its sequence of Fourier coefficients by calculating a *sum*. In the theory of the Fourier transform, we recover f from $\widehat{f}$ by calculating an integral. Namely, if g is any integrable function on the real line $\mathbb{R}$, then we define

$$g^{\vee}(t) = \int g(\xi) e^{2\pi i \xi \cdot t} \, d\xi. \tag{15.2.1.2}$$

The operation $^{\vee}$ is called the *inverse Fourier transform.*

It turns out that, whenever the integrals in question make sense, the operations $^{\vee}$ and $\widehat{\ }$ are inverse to each other. More precisely, if f is a function on the real line such that

- f is integrable,
- $\widehat{f}$ is integrable,

then

$$\left[\widehat{f}\right]^{\vee} = f. \tag{15.2.1.3}$$

An easily verified hypothesis that will guarantee that both f and $\widehat{f}$ are integrable is that $f \in C^2$ and f, f', f'' be integrable.

15.2.2 Some Fourier Transform Examples that Use Complex Variables

EXAMPLE 15.2.2.1 Let us calculate the Fourier transform of the function

$$f(t) = \frac{1}{1+t^2}. \tag{15.2.2.1.1}$$

Solution: The Fourier integral is

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-2\pi i t \cdot \xi} \, dt. \tag{15.2.2.1.2}$$

We will evaluate the integral using the calculus of residues.

For fixed ξ in $\mathbb{R}$, we thus consider the meromorphic function

$$m(z) = \frac{e^{-2\pi i z \cdot \xi}}{1+z^2}, \tag{15.2.2.1.3}$$

which has poles at $\pm i$.

For the case $\xi > 0$ it is convenient to use as contour of integration the positively oriented semicircle γ_R of radius $R > 1$ in the lower half-plane

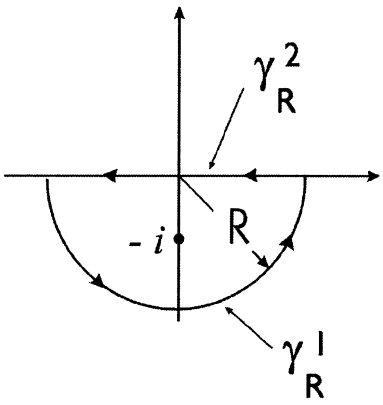


FIGURE 15.3
A positively oriented semicircle γ_R in the lower half-plane.

that is shown in Figure 15.3. Of course this contour only contains the pole at $-i$. We find that

$$\begin{aligned} 2\pi i \mathrm{Res}_m(-i) &= \int_{\gamma_R} m(z)\,dz \\ &= \int_{\gamma_R^1} m(z)\,dz + \int_{\gamma_R^2} m(z)\,dz. \end{aligned} \tag{15.2.2.1.4}$$

The integral over γ_R^1 vanishes as $R \to +\infty$ and the integral over γ_R^2 tends to

$$-\int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-2\pi i t\cdot\xi}\,dt. \tag{15.2.2.1.5}$$

It is straightforward to calculate that

$$\begin{aligned} \mathrm{Res}_m(-i) &= \lim_{z\to -i} (z-(-i)) \cdot \frac{e^{-2\pi i\xi z}}{z^2+1} \\ &= \frac{e^{-2\pi i\xi z}}{z-i}\bigg|_{z=-i} \\ &= \frac{e^{-2\pi i\xi(-i)}}{-2i} \\ &= \frac{e^{-2\pi\xi}}{-2i}. \end{aligned} \tag{15.2.2.1.6}$$

Thus

$$2\pi i \mathrm{Res}_m(-i) = -\pi e^{-2\pi\xi}. \tag{15.2.2.1.7}$$

We conclude that

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-2\pi i t\cdot\xi}\,dt = \pi e^{-2\pi\xi}. \tag{15.2.2.1.8}$$

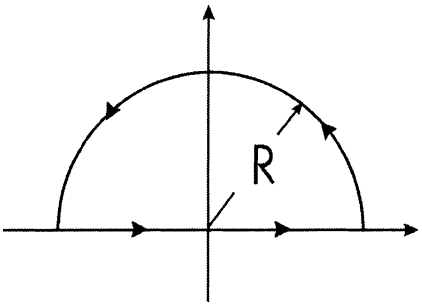


FIGURE 15.4
A positively oriented semicircle in the upper half-plane.

A similar calculation, using the contour μ_R shown in Figure 15.4, shows that, when $\xi < 0$, then

$$\widehat{f}(\xi) = \pi e^{2\pi\xi}. \tag{15.2.2.1.9}$$

In summary, for any $\xi \in \mathbb{R}$,

$$\widehat{f}(\xi) = \pi e^{-2\pi|\xi|}. \tag{15.2.2.1.10}$$

We can now check our work using the inverse Fourier transform: We observe that both f and $\widehat{f}$ are integrable, so we calculate that

$$\begin{aligned}
\left[\widehat{f}\right]^{\vee}(t) &= \int_{-\infty}^{\infty} \widehat{f} e^{2\pi i\xi\cdot t}\, d\xi \\
&= \int_0^{\infty} \pi e^{-2\pi\xi} e^{2\pi i\xi t}\, d\xi + \int_{-\infty}^{0} \pi e^{2\pi\xi} e^{2\pi i\xi t}\, d\xi \\
&= \int_0^{\infty} \pi e^{-2\pi\xi} e^{2\pi i\xi t}\, d\xi + \int_0^{\infty} \pi e^{-2\pi\xi} e^{-2\pi i\xi t}\, d\xi \\
&= 2\mathrm{Re}\left[\int_0^{\infty} \pi e^{-2\pi\xi} e^{2\pi i\xi t}\, d\xi\right] \\
&= 2\mathrm{Re}\left[\int_0^{\infty} \pi e^{\xi(-2\pi+2\pi it)}\, d\xi\right] \\
&= \mathrm{Re}\left[\frac{2\pi}{-2\pi+2\pi it}\cdot e^{\xi(-2\pi+2\pi it)}\right]_{\xi=0}^{\xi=\infty} \\
&= \mathrm{Re}\left[\frac{1}{-1+it}\cdot(0-1)\right] \\
&= \frac{1}{1+t^2}.
\end{aligned} \tag{15.2.2.1.11}$$

Observe that our calculations confirm the correctness of our Fourier transform determination. In addition, they demonstrate the validity of

the Fourier inversion formula in a particular instance. □

EXAMPLE 15.2.2.2 Physicists call a function of the form

$$f(t) = \begin{cases} \cos 2\pi t & \text{if } -7/4 \le t \le 7/4 \\ 0 & \text{if } t < -7/4 \text{ or } t > 7/4 \end{cases} \tag{15.2.2.2.1}$$

a *finite wave train.* Let us calculate the Fourier transform of this function.

Solution: The Fourier integral is

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi \cdot t}\, dt \\
&= \int_{-7/4}^{7/4} (\cos 2\pi t) e^{-2\pi i \xi \cdot t}\, dt \\
&= \frac{1}{2}\left[\int_{-7/4\pi}^{7/4\pi} e^{2\pi i t} e^{-2\pi i \xi \cdot t}\, dt + \int_{-7/4\pi}^{7/4\pi} e^{-2\pi i t} e^{-2\pi i \xi \cdot t}\, dt \right] \\
&= \frac{1}{2}\left[\int_{-7/4\pi}^{7/4\pi} e^{(2\pi i - 2\pi i \xi)t}\, dt + \int_{-7/4\pi}^{7/4\pi} e^{(-2\pi i - 2\pi i \xi)t}\, dt \right] \\
&= \frac{1}{2}\left(\left[\frac{1}{2\pi i - 2\pi i \xi} e^{(2\pi i - 2\pi i \xi)t} \right]_{t=-7/4}^{t=7/4} \right. \\
&\qquad \left. + \left[\frac{1}{-2\pi i - 2\pi i \xi} e^{(-2\pi i - 2\pi i \xi)t} \right]_{t=-7/4}^{t=7/4} \right) \\
&= \frac{1}{2}\left\{ \frac{1}{2\pi i(1-\xi)} \left[e^{2\pi i(1-\xi)[7/4]} - e^{2\pi i(1-\xi)[-7/4]} \right] \right. \\
&\qquad \left. + \frac{1}{2\pi i(-1-\xi)} \left[e^{-2\pi i(1+\xi)[7/4]} - e^{-2\pi i(1+\xi)[-7/4]} \right] \right\} \\
&= \frac{1}{2}\left\{ \frac{1}{2\pi i(1-\xi)} 2i \sin\left(\frac{7\pi}{2}(1-\xi) \right) \right. \\
&\qquad \left. - \frac{1}{2\pi i(-1-\xi)} 2i \sin\left(\frac{7\pi}{2}(1+\xi) \right) \right\} \\
&= \frac{1}{2\pi(1-\xi)} \left[\sin\frac{7\pi}{2} \cos\frac{7\pi}{2}\xi - \cos\frac{7\pi}{2} \sin\frac{7\pi}{2}\xi \right] \\
&\qquad - \frac{1}{2\pi(-1-\xi)} \left[\sin\frac{7\pi}{2} \cos\frac{7\pi}{2}\xi + \cos\frac{7\pi}{2} \sin\frac{7\pi}{2}\xi \right] \\
&= \frac{1}{2\pi(1-\xi)} \left(-\cos\frac{7\pi}{2}\xi \right) + \frac{1}{2\pi(1+\xi)} \left(-\cos\frac{7\pi}{2}\xi \right)
\end{aligned}$$

$$= \frac{1}{2\pi} \cos\frac{7\pi}{2}\xi \left\{ \frac{-1}{1+\xi} - \frac{1}{1-\xi} \right\}$$
$$= -\frac{1}{\pi}\left(\cos\frac{7\pi}{2}\xi\right)\frac{1}{1-\xi^2}. \qquad (15.2.2.2.2)$$

In summary,

$$\widehat{f}(\xi) = -\frac{1}{\pi}\cos\left(\frac{7\pi}{2}\xi\right)\frac{1}{1-\xi^2}. \qquad (15.2.2.2.3)$$

We may now perform a calculation to confirm the Fourier inversion formula in this example. The calculus of residues will prove to be a useful tool in the process.

Now

$$\begin{aligned}
\left[\widehat{f}\right]^{\vee}(t) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1-\xi^2}\cos\frac{7\pi}{2}\xi \cdot e^{2\pi i\xi t}\,d\xi \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1-\xi^2}\left[e^{(7\pi/2)\xi i} + e^{-(7\pi/2)\xi i}\right] e^{2\pi i\xi t}\,d\xi \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1-\xi^2} e^{i\xi[(7\pi/2)+2\pi t]}\,d\xi \\
&\qquad -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1-\xi^2} e^{i\xi[(-7\pi/2)+2\pi t]}\,d\xi \\
&\equiv I + II. \qquad (15.2.2.2.4)
\end{aligned}$$

First we analyze expression I. For $t \geq -7/4$ fixed, the expression $(7\pi/2) + 2\pi t$ is non-negative. Thus the exponential expression will be bounded (i.e., the real part of the exponent will be non-positive), if we integrate the meromorphic function

$$\frac{1}{1-z^2} e^{iz[(7\pi/2)+2\pi t]} \qquad (15.2.2.2.5)$$

on the curve $\gamma_{r,R}$ exhibited in Figure 15.5. It is easy to see that the integral over $\gamma^1_{r,R}$ tends to zero as $R \to +\infty$. And the integrals over $\gamma^2_{r,R}, \gamma^4_{r,R}, \gamma^6_{r,R}$ tend to the integral that is I as $r \to 0$. It remains to evaluate the integrals over $\gamma^3_{r,R}$ and $\gamma^5_{r,R}$. We do the first and leave the second for the reader.

Now

$$\begin{aligned}
&-\frac{1}{2\pi}\oint_{\gamma^3_{r,R}} \frac{1}{1-z^2} e^{iz[(7\pi/2)+2\pi t]}\,dz \\
&= -\frac{1}{2\pi}\int_{\pi}^{0} \frac{1}{1-(-1+re^{i\theta})^2} e^{i(-1+re^{i\theta})[(7\pi/2)+2\pi t]} ire^{i\theta}\,d\theta \\
&= -\frac{1}{2\pi}\int_{\pi}^{0} \frac{ire^{i\theta}}{2re^{i\theta} - r^2e^{2i\theta}} e^{i(-1+re^{i\theta})[(7\pi/2)+2\pi t]}\,d\theta
\end{aligned}$$

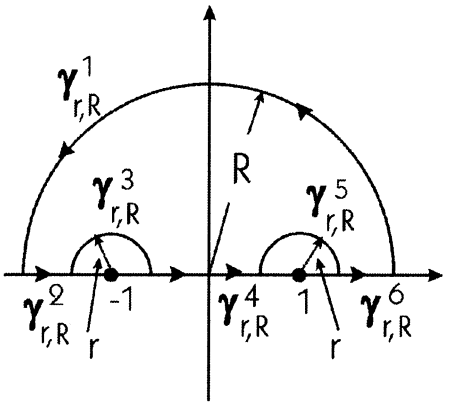


FIGURE 15.5
The curve $\gamma_{r,R}$.

$$
\begin{aligned}
&= -\frac{i}{2\pi}\int_{\pi}^{0} \frac{1}{2-re^{i\theta}} e^{i(-1+re^{i\theta})[(7\pi/2)+2\pi t]}\, d\theta \\
&\overset{(r\to 0)}{\longrightarrow} -\frac{i}{2\pi}\int_{\pi}^{0} \frac{1}{2} e^{-i[(7\pi/2)+2\pi t]}\, d\theta \\
&= -\frac{1}{4} e^{-2\pi i t}. \qquad (15.2.2.2.6)
\end{aligned}
$$

A similar calculation shows that

$$
-\frac{1}{2\pi}\oint_{\gamma^5_{r,R}} \frac{1}{1-z^2} e^{iz[(7\pi/2)+2\pi t]}\, dz = -\frac{1}{4} e^{2\pi i t}. \qquad (15.2.2.2.7)
$$

In sum,

$$
I = \frac{1}{2\pi}\oint_{\gamma^3_{r,R}} + \frac{1}{2\pi}\oint_{\gamma^5_{r,R}} = -\frac{1}{2}\cos 2\pi t. \qquad (15.2.2.2.8)
$$

To analyze the integral II, we begin by fixing $t > 7/4$. Then we will have $(-7\pi/2) + 2\pi t > 0$. If we again use the contour in Figure 15.5, then the exponential in the meromorphic function

$$
\frac{1}{1-z^2} e^{iz[(-7\pi/2)+2\pi t]} \qquad (15.2.2.2.9)
$$

will be bounded on the curve $\gamma^1_{r,R}$. Of course the integral over $\gamma^1_{r,R}$ tends to zero as $R \to +\infty$. The integrals over $\gamma^2_{r,R}, \gamma^4_{r,R}, \gamma^6_{r,R}$ in sum tend to the

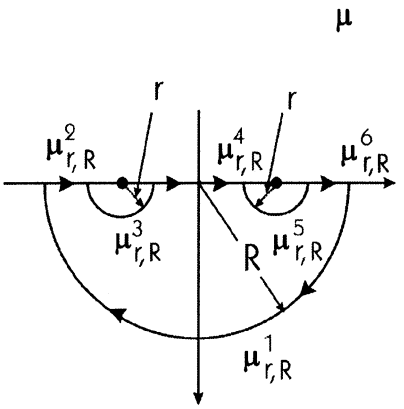


FIGURE 15.6
The curve $\mu_{r,R}$.

integral that defines II. Finally, a calculation that is nearly identical to the one that we just performed for I shows that

$$-\frac{1}{2\pi}\oint_{\gamma^3_{r,R}} \frac{1}{1-z^2} e^{iz[(-7\pi/2)+2\pi t]}\, dz = \frac{1}{4}e^{-2\pi i t}. \tag{15.2.2.2.10}$$

Similarly,

$$-\frac{1}{2\pi}\oint_{\gamma^5_{r,R}} \frac{1}{1-z^2} e^{iz[(-7\pi/2)+2\pi t]}\, dz = \frac{1}{4}e^{-2\pi i t}. \tag{15.2.2.2.11}$$

Therefore

$$II = \frac{1}{2}\cos 2\pi t. \tag{15.2.2.2.12}$$

In summary, we see that on the common domain $t > 7/4$ we have

$$I + II = -\frac{1}{2}\cos 2\pi t + \frac{1}{2}\cos 2\pi t = 0. \tag{15.2.2.2.13}$$

This value agrees with $f(t)$ when $t > 7/4$.

Similar calculations for $t < -7/4$ (but using the contour shown in Figure 15.6) show that

$$\left[\widehat{f}\right]^{\vee}(t) = 0. \tag{15.2.2.2.14}$$

The remaining, and most interesting, calculation is for $-7/4 \leq t \leq 7/4$. We have already calculated I for that range of t. To calculate II, we use the contour in Figure 15.6. The result is that

$$II = -\frac{1}{2}\cos 2\pi t. \tag{15.2.2.2.15}$$

Therefore

$$[\widehat{f}\,]^{\vee}(t) = -(I + II) = \frac{1}{2}\cos 2\pi t + \frac{1}{2}\cos 2\pi t = \cos 2\pi t. \quad (15.2.2.2.16)$$

This confirms Fourier inversion for the finite wave train. □

15.2.3 Solving a Differential Equation Using the Fourier Transform

A simple integration by parts confirms that if $f \in C^1(\mathbb{R})$ and if both f and f' are integrable, then

$$\widehat{f'}(\xi) = 2\pi i \xi \widehat{f}(\xi). \quad (15.2.3.1.1)$$

This formula is elementary but important. It is analogous to formula (15.1.6.5) for Fourier series coefficients. We can use it to solve a differential equation:

EXAMPLE 15.2.3.2 Use the Fourier transform to solve the differential equation

$$f''(t) - f(t) = \varphi(t), \quad (15.2.3.2.1)$$

where

$$\varphi(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \quad (15.2.3.2.2)$$

Solution: We begin by applying the Fourier transform to both sides of equation (15.2.3.2.1). The result is

$$-4\pi^2\xi^2\widehat{f}(\xi) - \widehat{f}(\xi) = \widehat{\varphi}. \quad (15.2.3.2.3)$$

An easy calculation shows that

$$\widehat{\varphi}(\xi) = -\frac{1}{1 + 2\pi i \xi}. \quad (15.2.3.2.4)$$

Thus, under the Fourier transform, our ordinary differential equation has become

$$-4\pi^2\xi^2\widehat{f}(\xi) - \widehat{f}(\xi) = -\frac{1}{1 + 2\pi i \xi} \quad (15.2.3.2.5)$$

or

$$\widehat{f}(\xi) = \frac{1}{(4\pi^2\xi^2 + 1)(1 + 2\pi i \xi)}. \quad (15.2.3.2.6)$$

Of course the expression on the right-hand side of (15.2.3.2.6) has no singularities on the real line (thanks to complex variables) and is integrable.

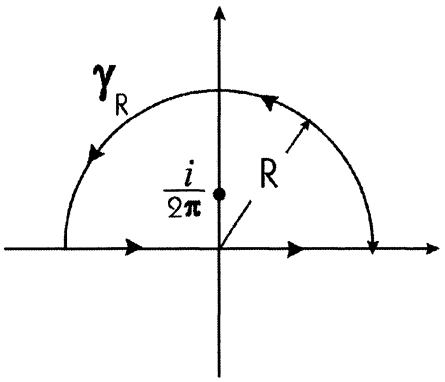


FIGURE 15.7
The contour used to analyze the integral in (15.2.3.2.8).

So we may apply the Fourier inversion formula (§15.2.1) to both sides of (15.2.3.2.6) to obtain

$$f(t) = [\widehat{f}]^{\vee}(t) = \left[\frac{1}{(4\pi^2\xi^2 + 1)(1 + 2\pi i\xi)}\right]^{\vee}. \tag{15.2.3.2.7}$$

We can find the function f if we can evaluate the expression on the right-hand side of (15.2.3.2.7). This amounts to calculating the integral

$$\int_{-\infty}^{\infty} \frac{1}{(4\pi^2\xi^2 + 1)(1 + 2\pi i\xi)} e^{2\pi i\xi t}\, d\xi. \tag{15.2.3.2.8}$$

We will do so for $t > 0$ (the most interesting set of values for t, given the data function φ in the differential equation) and let the reader worry about $t \leq 0$.

It is helpful to use the calculus of residues to evaluate the integral in (15.2.3.2.8). We use the contour in Figure 15.7, chosen (with $R >> 1$) so that the exponential in the integrand will be bounded when the variable is on the curve and $t > 0$. The pole of

$$m(z) = \frac{1}{(4\pi^2 z^2 + 1)(1 + 2\pi iz)} e^{2\pi izt} = \frac{1}{(1 - 2\pi iz)(1 + 2\pi iz)^2} e^{2\pi izt} \tag{15.2.3.2.9}$$

that lies inside of γ_R is at $P = i/[2\pi]$. This is a pole of order two. We use the formula in §§4.4.6. The result is that

$$\mathrm{Res}_m(P) = \frac{-i}{4\pi}\left[\frac{1}{2}e^{-t} + te^{-t}\right]. \tag{15.2.3.2.10}$$

As usual, the value of the integral

$$\int_{-\infty}^{\infty} \frac{1}{(4\pi^2\xi^2+1)(1+2\pi i\xi)} e^{2\pi i\xi t}\, d\xi \tag{15.2.3.2.11}$$

is

$$(-2\pi i)\cdot\frac{-i}{4\pi}\left[\frac{1}{2}e^{-t} + te^{-t}\right] = -\frac{1}{2}\left[\frac{1}{2}e^{-t} + te^{-t}\right]. \tag{15.2.3.2.12}$$

This is the solution f (at least when $t > 0$) of our differential equation (15.2.3.2.1). □

15.3 The Laplace Transform

15.3.1 Prologue

Let f be an integrable function on the half-line $\{t \in \mathbb{R} : t \geq 0\}$. [We implicitly assume that $f(t) = 0$ when $t < 0$.]

In many contexts, it is convenient to think of the Fourier transform

$$\widehat{f}(\xi) = \int f(t)e^{-2\pi i\xi\cdot t}\, dt \tag{15.3.1.1}$$

as a function of the complex variable ξ. In fact when $\mathrm{Im}\,\xi < 0$ and $t > 0$ the exponent in the integrand has negative real part so the exponential is bounded and the integral converges. For suitable f one can verify, using Morera's theorem (§§2.3.2) for instance, that $\widehat{f}(\xi)$ is a *holomorphic* function of ξ. It is particularly convenient to let ξ be pure imaginary: the customary notation is $\xi = -is$ for $s \geq 0$. Then we have defined a new function

$$F(s) = \int f(t)e^{-st}\, dt. \tag{15.3.1.2}$$

(As is customary in the theory of the Laplace transform, we have omitted the factor of 2π.) We call F the *Laplace transform* of f. Sometimes, instead of writing F, we write $\mathcal{L}(f)$.

The Laplace transform is a useful tool because

- It has formal similarities to the Fourier transform.
- It can be applied to a larger class of functions than the Fourier transform (since the fact e^{-st} decays rapidly at infinity).
- It is often straightforward to compute.

The lesson here is that it is sometimes useful to modify a familiar mathematical operation (in this case the Fourier transform) by letting the variable be complex (in this case producing the Laplace transform).

We now provide just one example to illustrate the utility of the Laplace transform.

15.3.2 Solving a Differential Equation Using the Laplace Transform

EXAMPLE 15.3.2.1 Use the Laplace transform to solve the ordinary differential equation

$$f''(t) + 3f'(t) + 2f(t) = \sin t \tag{15.3.2.1.1}$$

subject to the initial conditions $f(0) = 1$, $f'(0) = 0$.

Solution: Working by analogy with Example 15.2.3.2, we calculate the Laplace transform of both sides of the equation. Integrating by parts (as we did when studying the Fourier transform—§§15.2.3), we can see that

$$\mathcal{L}(f')(s) = s \cdot \mathcal{L}f(s) - 1 \tag{15.3.2.1.2}$$

and

$$\mathcal{L}(f'')(s) = s^2\mathcal{L}f(s) - s. \tag{15.3.2.1.3}$$

(These formulas are correct when all the relevant integrals converge. See also the Table of Laplace Transforms on page 271.)

Thus equation (15.3.1.1) is transformed to

$$[s^2\mathcal{L}f(s) - s] + 3[s \cdot \mathcal{L}f(s) - 1] + 2\mathcal{L}f(s) = \big(\mathcal{L}[\sin t]\big)(s). \tag{15.3.2.1.4}$$

A straightforward calculation (using either integration by parts or complex variable methods) shows that

$$\mathcal{L}[\sin t](s) = \frac{1}{s^2 + 1}. \tag{15.3.2.1.5}$$

So equation (15.3.2.1.4) becomes

$$[s^2\mathcal{L}f(s) - s] + 3[s \cdot \mathcal{L}f(s) - 1] + 2\mathcal{L}f(s) = \frac{1}{s^2 + 1}. \tag{15.3.2.1.6}$$

Just as with the Fourier transform, the Laplace transform has transformed the differential equation to an algebraic equation. We find that

$$\mathcal{L}f(s) = \frac{1}{s^2+3s+2} \cdot \left[\frac{1}{s^2+1} + s + 3\right]. \tag{15.3.2.1.7}$$

We use the method of partial fractions to break up the right-hand side into simpler components. The result is

$$\mathcal{L}f(s) = \frac{5/2}{s+1} + \frac{-6/5}{s+2} + \frac{-3s/10+1/10}{s^2+1}. \tag{15.3.2.1.8}$$

Now our job is to find the inverse Laplace transform of each expression on the right. One way to do this is by using the Laplace inversion formula

$$f(t) = \int_{-i\infty}^{\infty} F(s)e^{st}\,ds. \tag{15.3.2.1.9}$$

However, the most common method is to use a Table of Laplace Transforms, as in [SASN, p. 402] or [CRC, pp. 559-564] or the Table of Laplace transforms on page 273 of this book. From such a table, we find that

$$f(t) = \frac{5}{2}e^{-t} - \frac{6}{5}e^{-2t} - \frac{3}{10}\cos t + \frac{1}{10}\sin t. \tag{15.3.2.1.10}$$

The reader may check that this is indeed a solution to the differential equation (15.3.2.1.1) . □

15.4 The z-Transform

The z-transform, under that particular name, is more familiar in the engineering community than in the mathematics community. Mathematicians group this circle of ideas with the notion of generating function and with allied ideas from finite and combinatorial mathematics (see, for instance [STA]). Here we give a quick introduction to the z-transform and its uses.

15.4.1 Basic Definitions

Let $\{a_j\}_{j=-\infty}^{+\infty}$ be a doubly infinite sequence. The z-transform of this sequence is defined to be the series

$$A(z) = \sum_{j=-\infty}^{\infty} a_j z^{-j}. \tag{15.4.1.1}$$

If this series converges on some annulus centered at the origin, then of course it defines a holomorphic function. Often the properties of the original sequence $\{a_j\}_{j=-\infty}^{+\infty}$ can be studied by way of the holomorphic function A.

The reference [CRC, pp. 231, 543] explains the relationship between the z-transform and other transforms that we have discussed. The reference [HEN, v. 2, pp. 322, 327, 332, 334, 335, 336, 350] gives further instances of the technique of the z-transform.

15.4.2 Population Growth by Means of the z-Transform

We present a typical example of the use of z-series.

EXAMPLE 15.4.2.1 During a period of growth, a population of salmon has the following two properties:

(15.4.2.1.1) The population, on average, reproduces at the rate of 3% per month.

(15.4.2.1.2) One hundred new salmon swim upstream and join the population each month.

If $a(n)$ is the population in month n, then find a formula for $a(n)$.

Solution: Let P denote the initial population. Then we may describe the sequence in this recursive manner:

$$\begin{aligned} a(0) &= P \\ a(1) &= a(0) \cdot (1 + .03) + 100 \\ a(2) &= a(1) \cdot (1 + .03) + 100 \\ &\text{etc.} \end{aligned} \tag{15.4.2.1.3}$$

Because we are going to use the theory of the z-transform, it is convenient to postulate that $a(n) = 0$ for $n < 0$.

Let us assume that $\{a(n)\}$ has a z-transform $A(z)$—at least when z is sufficiently large. It is also convenient to think of each part of the recursion as depending on n. So let us set

$$P(n) = \begin{cases} P \text{ if } n = -1 \\ 0 \text{ if } n \neq -1 \end{cases} \tag{15.4.2.1.4}$$

and

$$s(n) = \begin{cases} 100 \text{ if } n \geq 0 \\ 0 \quad \text{if } n < 0. \end{cases} \tag{15.4.2.1.5}$$

Then our recursion can be expressed as

$$a(n+1) = 1.03a(n) + P(n) + s(n). \tag{15.4.2.1.6}$$

We multiply both sides of this equation by z^{-n} and sum over n to obtain

$$\sum_n a(n+1)z^{-n} = 1.03\sum_n a(n)z^{-n} + \sum_n P(n)z^{-n} + \sum_n s(n)z^{-n} \tag{15.4.2.1.7}$$

or

$$z \cdot A(z) = 1.03A(z) + Pz + \frac{100z}{z-1}. \tag{15.4.2.1.8}$$

Here, for the last term, we have used the elementary fact that

$$\sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty}(1/z)^n = \frac{1}{1-1/z} = \frac{z}{z-1}, \tag{15.4.2.1.9}$$

valid for $|z| > 1$.

Rearranging equation (15.4.2.1.8), we find that

$$A(z) = \frac{Pz^2 + (100-P)z}{(z-1)(z-1.03)} = z \cdot \frac{Pz + (100-P)}{(z-1)(z-1.03}, \tag{15.4.2.1.10}$$

valid for $|z|$ sufficiently large.

Of course we may decompose this last expression for $A(z)$ into a partial fractions decomposition:

$$A(z) = z \cdot \left[\frac{P + 100/.03}{z-1.03} - \frac{100/.03}{z-1}\right]. \tag{15.4.2.1.11}$$

We rewrite the terms in preparation of making a Laurent expansion:

$$A(z) = \left(P + \frac{100}{.03}\right) \cdot \left[\frac{1}{1-1.03/z}\right] - \frac{100}{.03} \cdot \frac{1}{1-1/z}. \tag{15.4.2.1.12}$$

For $|z| > 1.03$, we may use the standard expansion

$$\frac{1}{1-\alpha} = \sum_{n=0}^{\infty} \alpha^n, \qquad |\alpha| < 1 \tag{15.4.2.1.13}$$

to obtain

$$A(z) = \left(P + \frac{100}{.03}\right)\sum_{n=0}^{\infty}(1.03)^n z^{-n} - \frac{100}{.03}\sum_{n=0}^{\infty} z^{-n}. \tag{15.4.2.1.14}$$

Note that we have obtained the expansion of $A(z)$ as a z-series! Its coefficients must therefore be the $a(n)$. We conclude that

$$a(n) = \begin{cases} \left(P + \frac{100}{.03}\right)(1.03)^n - \frac{100}{.03} & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases} \qquad (15.4.2.1.15)$$

It is easy to see that this problem could have been solved without the aid of the z-transform. But the z-transform was a useful device for keeping track of information. □

Chapter 16

Computer Packages for Studying Complex Variables

16.0 Introductory Remarks

In the past decade or so, there has been a wide proliferation of high speed digital computing equipment. Concomitant with that growth has been the development of ever more sophisticated analytic tools for doing mathematics. Gone are the days of using the computer simply as a "number cruncher." Now there are sophisticated computer algebra software and two- and three-dimensional graphing software. In the present chapter we shall give brief descriptions of the major packages that are useful in doing complex analysis.

We note that the producers of computer algebra systems in general have been slow to respond to the need for complex-analytic computing capabilities. It is only the latest releases of `Mathematica` and `Maple` that have serious power in complex arithmetic. The specialized software `f(z)` does *only* complex analysis, but its capabilities are very particular.

16.1 The Software Packages

16.1.1 The Software f(z)®

The software `f(z)`®, available for both the PC and Macintosh platforms, was one of the first to accept complex-analytic input and to be able to produce graphs of holomorphic functions. The most recent release for the Windows95 platform, available on CD-ROM, is particularly attractive.

The primary purpose of `f(z)`® is to graph holomorphic functions. Of

S. G. Krantz, *Handbook of Complex Variables*
© Springer Science+Business Media New York 1999

course the graph actually lives in $\mathbb{C} \times \mathbb{C}$, which is 4-dimensional Euclidean space. The software f(z)® offers the user the option of viewing the graph in a great variety of configurations. Among these are

(16.1.1.1) As a collection of images (in the plane) of circular level curves.

(16.1.1.2) As a collection of images (in the plane) of rectilinear level curves.

(16.1.1.3) Either **(16.1.1.1)** or **(16.1.1.2)** with the images lying in the Riemann sphere.

(16.1.1.4) As a graph in three dimensions (where attention is restricted to either the real part or the imaginary part of the image).

(16.1.1.5) As a graph in 4-dimensional space.

Of course **(16.1.1.5)** can only be suggested through a variety of graphical tricks. In particular, animations are used in a compelling manner to suggest how various three-dimensional cross-sections fit together to compose the four-dimensional "graph." The commands in f(z)® enable easy redefinition of the function, zooming, repositioning, and rotation.

Functions may be defined by direct entry of the function name or by composition or by iteration. It is possible to draw Julia sets, fractals, Mandelbrot sets, and the various states generated by the complex Newton's method.

Beautiful line printouts are straightforward to produce; resolution, colors, and other attributes may be adjusted to suit. The software is transparent to use; resort to the documentation is rarely necessary.

Although the strength of this software has traditionally been graphics, the newest Windows95 release also features numerical computation of complex line integrals. One can calculate winding numbers and verify the Cauchy integral formula and theorem in particular instances.

Figures 16.1–16.12 exhibit some representative f(z)® output. They depict the following:

[**The function $f(z) = e^z$**] Figure 16.1 shows the image in the plane of circular level curves under e^z; Figure 16.2 shows the image in the Riemann sphere of circular level curves under e^z; Figure 16.3 shows the graph in 3-space of circular level curves under e^z.

[**The function $f(z) = z^2$**] Figure 16.4 shows the image in the plane of circular level curves under z^2; Figure 16.5 shows the image in the Riemann sphere of circular level curves under z^2; Figure 16.6 shows the graph in 4-space of circular level curves acted on by z^2.

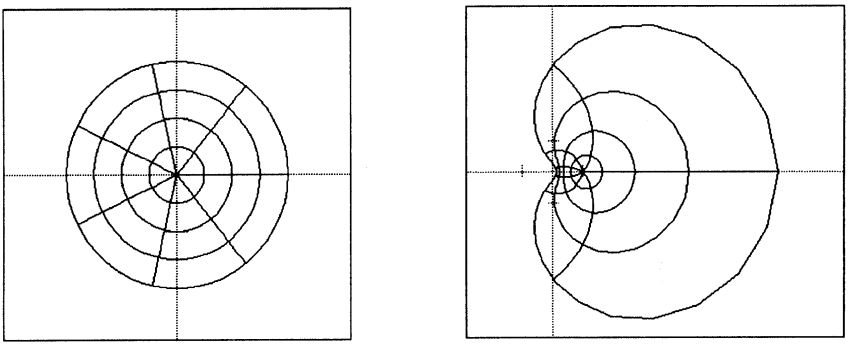


FIGURE 16.1
The image in the plane of circular level curves under e^z.

[Image of a rectangular grid] Figure 16.7 shows the image in the plane of a rectangular grid under e^z; Figure 16.8 shows the image in the Riemann sphere of a rectangular grid under z^2; Figure 16.9 the graph in 4-space of a rectangular grid under $z^3 - 3z^2 + z - 2$.

[The image of a rectangular grid under $f(z) = \log z$] Figure 16.10 shows the image in the plane of a rectangular grid under $\log z$; Figure 16.11 shows the image in the Riemann sphere of a rectangular grid under $\log z$; Figure 16.12 shows the graph in 4-space of a rectangular grid under $\log z$.

16.1.2 Mathematica®

Mathematica® is a powerful all-around mathematics utility. It can perform computer algebra operations and numerical calculations, and has stunning graphing utilities. Version 3 is particularly well-equipped with complex analysis capabilities. See [WOL] for details of the Mathematica® syntax.

A complex number is denoted `a + bI`. Mathematica® can convert such a complex number to polar form, expressed either with trigonometric functions or in exponential notation. All complex arithmetic operations are performed with straightforward commands. For example, the input

```
(4 + 3I)/(2 - I)                                    (16.1.2.1)
```

yields the output `1 + 2I`. The commands `Re[z]`, `Im[z]`, `Conjugate[z]`, `Abs[z]`, `Arg[z]` compute real part, imaginary part, conjugate, modulus, and argument, respectively. (Continued on page 227.)

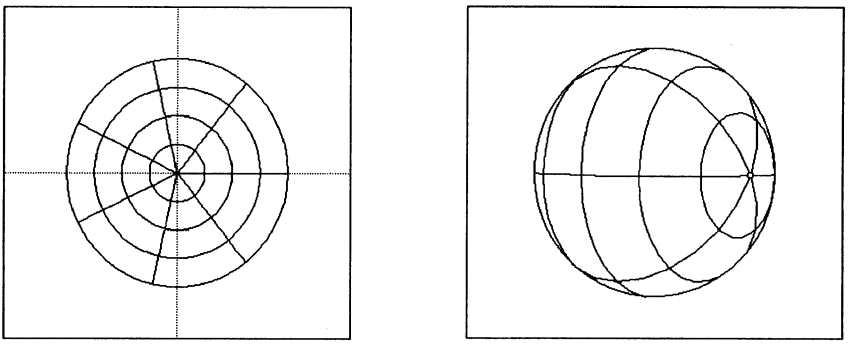


FIGURE 16.2
The image in the Riemann sphere of circular level curves under e^z.

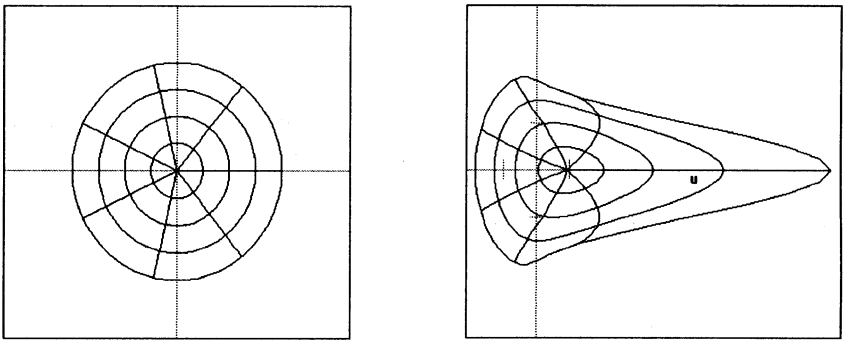


FIGURE 16.3
The graph in 3-space of circular level curves acted on by e^z.

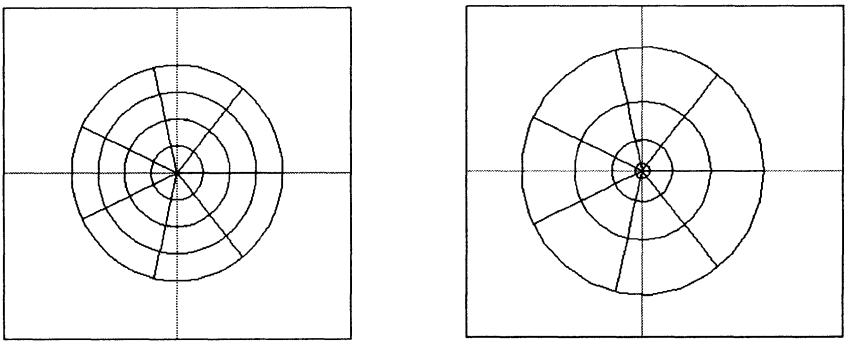


FIGURE 16.4

The image in the plane of circular level curves under z^2.

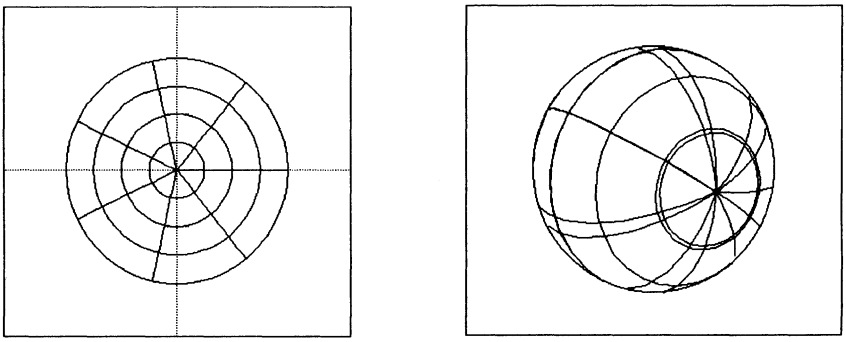


FIGURE 16.5
The image in the Riemann sphere of circular level curves under z^2.

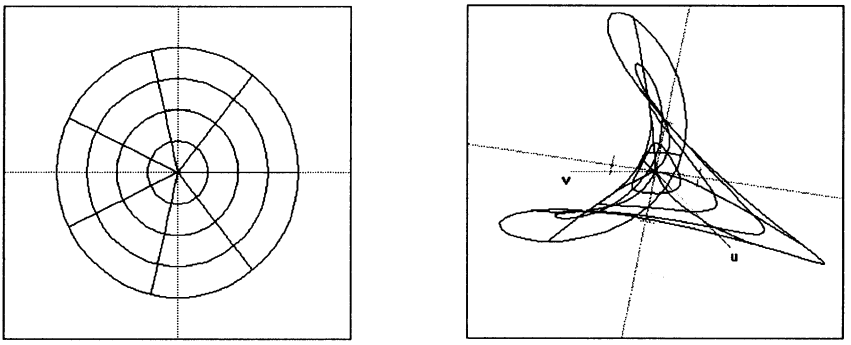


FIGURE 16.6
The graph in 4-space of circular level curves acted on by z^2.

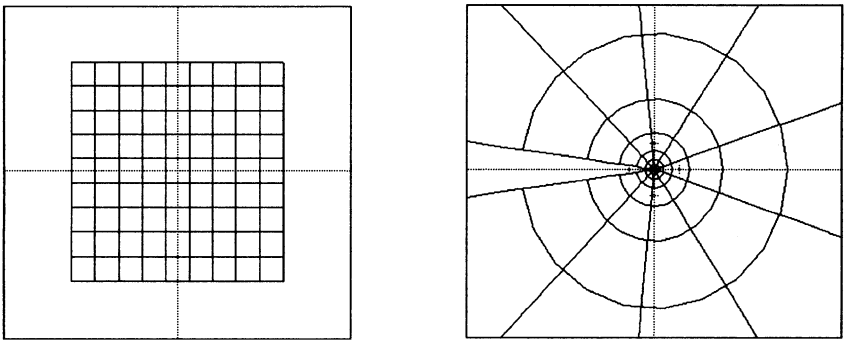


FIGURE 16.7
The image in the plane of a rectangular grid under e^z.

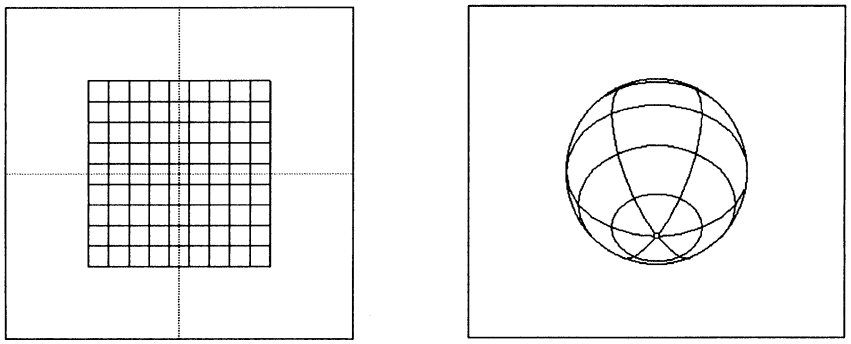


FIGURE 16.8
The image in the Riemann sphere of a rectangular grid under z^2.

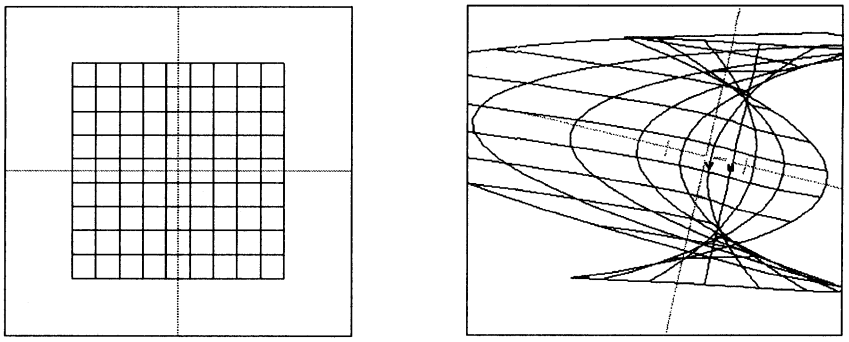


FIGURE 16.9
The graph in 4-space of a rectangular grid acted on by $z^3 - 3z^2 + z - 2$.

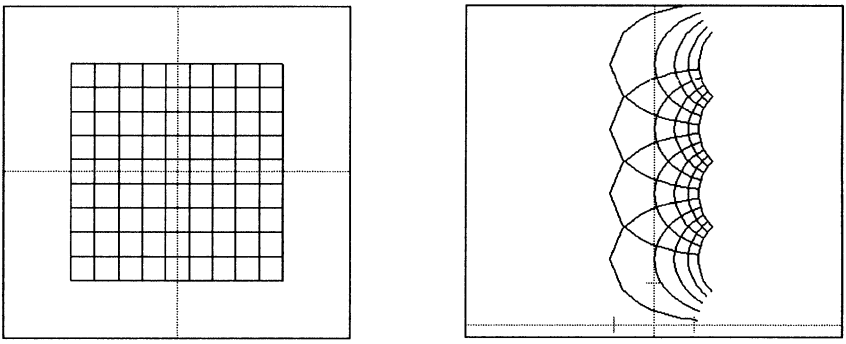


FIGURE 16.10
The image in the plane of a rectangular grid under $\log z$.

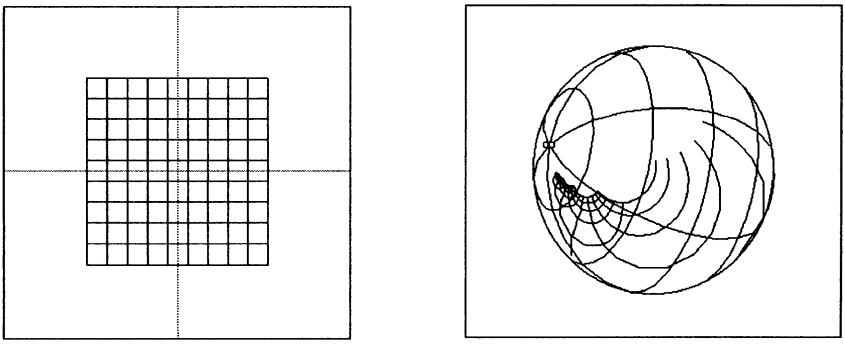


FIGURE 16.11
The image in the Riemann sphere of a rectangular grid under $\log z$.

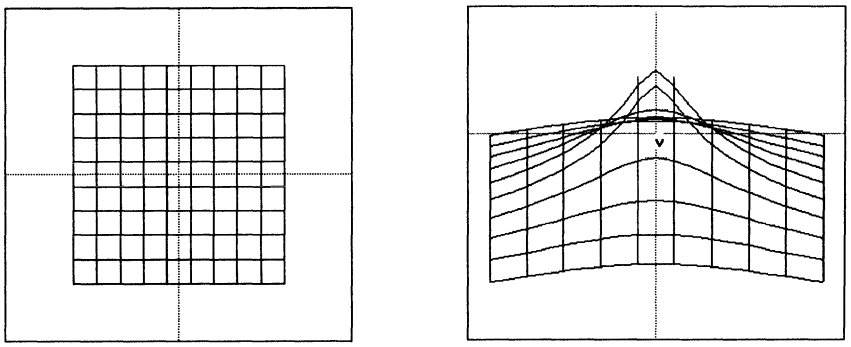


FIGURE 16.12
The graph in 4-space of a rectangular grid acted on by $\log z$.

`Mathematica`® can factor a polynomial over the Gaussian integers: for example, the command

```
Factor[1 + x^2, GaussianIntegers -> True]
```
(16.1.2.2)

yields the output `(-I + x)(I + x)`.

`Mathematica`® can calculate residues. The input

```
Residue[1/x, {x,0}]
```
(16.1.2.3)

yields the output `1`, while the input

```
Residue[1/x^2, {x,0}]
```
(16.1.2.4)

yields the output `0`.

`Mathematica`® has a number of specialized commands for converting a (complex) mathematical expression from one scientific form to another. Some examples are in the Table of `Mathematica`® commands.

16.1.3 `Maple`®

`Maple`® is another all-around mathematics utility. Like `Mathematica`®, it can perform computer algebra operations and run numerical analysis routines; it also has powerful graphics capabilities.

In general, `Maple`®'s capabilities are similar to those of `Mathematica`.® Of course the syntax is different. Details of `Maple`® syntax may be found in [CHA].

`Maple`® has a command `evalc` that "oversees" a number of basic complex arithmetic operations. The Table of `Maple`® commands exhibits some uses of `evalc` and the corresponding output:

Examples of Mathematica® Commands

input	output
`Exp[2 + 9I]//N`	`-6.73239 + 3.04517 I`
`ComplexExpand[Sin[x + Iy]]`	`Cosh[y] Sin [x] + I Cos[x] Sinh [y]`
`TrigToExp[Tan[x]]`	$[I(E^{-Ix} - E^{Ix})]/[E^{-Ix} + E^{Ix}]$
`ExpToTrig[Exp[x] - Exp[-x]]`	`2 Sinh[x]`
`N[Sqrt[-2I]]`	`1. - 1.I`

Examples of Maple® Commands

input	output
`evalc((3 + 5*I)*(7 + 4*I));`	`1 + 47 I`
`evalc(Re((1 + 2*I)*(3 - 4*I)));`	`11`
`evalc((-5 + 7*I)/(2 + 3*I));`	`1 + 2*I`
`evalc(sin(I));`	`sinh(1) I`
`evalc(exp(I));`	`cos(1) + sin(1) I`
`evalc(conjugate(exp(2*I)));`	`cos(2) - sin(2) I`
`evalc(polar(r,theta));`	`r cos(theta) + r sin(theta) I`

A feature of many of the most popular computer algebra systems is that the user must *tell* the software when complex arithmetic is desired. The command `evalc` is an illustration of that concept.

Maple® can convert a complex number from Cartesian form to polar form, expressed either with exponentials or with trigonometric functions.

Using the command `fsolve` with the `complex` option, Maple® can find all the roots of any polynomial with complex coefficients; the roots are expressed as floating point complex numbers.

Maple® can compute the signum of a complex number where $\operatorname{sgn} z$ is defined to be $z/|z|$ when $z \neq 0$. The command is `signum z`.

Maple® also has advanced capabilities, such as being able to handle the syntax of the Newman-Penrose conjugation operator. It can treat Hermitian tensors and spinors.

16.1.4 MatLab®

The package MatLab® has the Maple® kernel imbedded in it. But it has MatLab®'s powerful front end. Therefore, with MatLab,® one can perform any complex arithmetic operation that can be performed in Maple.® See [HAL], [MAT], [MOC] for details of the MatLab® syntax.

An attractive feature of MatLab® is that complex numbers do not require special treatment. They can be entered (using either i or j for $\sqrt{-1}$) just as one would enter a real number.

16.1.5 Ricci®

Ricci® is a Mathematica® package created by John M. Lee. It is available, together with descriptive material, add-ons, and documentation, from the Web site

`\tt http://www.math.washington.edu/~lee/Ricci/`

Ricci® is designed to do tensor calculations in differential geometry. As such, it is tangential to the main thrust of the present book. For those who know the formalism of differential geometry, it may be of interest to know that Ricci® can calculate covariant derivatives, exterior derivatives, Riemannian metrics and curvatures, can manipulate vector bundles, can handle complex bundles and tensors, and conforms to the Einstein summation convention.

When performed by hand, the calculations described in the preceding paragraph are massive—and easily prone to error. Ricci® is a powerful device for accuracy checking and for performing "what if" experiments in differential geometry. We note that it is *not* a commercial product.

Glossary of Terms from Complex Variable Theory and Analysis

accumulation point Let $a_1, a_2, \ldots$ be points in the complex plane. A point b is an accumulation point of the sequence $\{a_j\}$ if the a_j get arbitrarily close to b. More formally, we require that for each $\epsilon > 0$ there exists an $N > 0$ such that when $j > N$, then $|a_j - b| < \epsilon$. §§3.2.1.

analytic continuation The procedure for enlarging the domain of a holomorphic function. §§10.1.1, §§10.1.2.

analytic continuation of a function If $(f_1, U_1), \ldots, (f_k, U_k)$ are function elements and if each (f_j, U_j) is a direct analytic continuation of (f_{j-1}, U_{j-1}), $j = 2, \ldots, k$, then we say that (f_k, U_k) is an *analytic continuation* of (f_1, U_1). §§10.1.5.

analytic continuation of a function element along a curve An *analytic continuation* of (f, U) along the curve γ is a collection of function elements (f_t, U_t), $t \in [0, 1]$, such that

1) $(f_0, U_0) = (f, U)$.

2) For each $t \in [0, 1]$, the center of the disc U_t is $\gamma(t)$, $0 \leq t \leq 1$.

3) For each $t \in [0, 1]$, there is an $\epsilon > 0$ such that, for each $t' \in [0, 1]$ with $|t' - t| < \epsilon$, it holds that

   (a) $\gamma(t') \in U_t$ and hence $U_{t'} \cap U_t \neq \emptyset$;

   (b) $f_t \equiv f_{t'}$ on $U_{t'} \cap U_t$ [so that (f_t, U_t) is a direct analytic continuation of $(f_{t'}, U_{t'})$]. §§10.2.1.

annulus A set of one of the forms $\{z \in \mathbb{C} : 0 < |z| < R\}$ or $\{z \in \mathbb{C} : r < |z| < R\}$ or $\{z \in \mathbb{C} : r < |z| < \infty\}$. §§4.2.3.

area principle If f is schlicht and if

$$h(z) = \frac{1}{f(z)} = \frac{1}{z} + \sum_{j=0}^{\infty} b_j z^j,$$

then

$$\sum_{j=1}^{\infty} j|b_j|^2 \leq 1. \qquad \text{§§12.1.4}$$

argument If $z = re^{i\theta}$ is a complex number written in polar form, then θ is the argument of z. §§1.2.6.

argument principle Let f be a function that is holomorphic on a domain that contains the closed disc $\overline{D}(P,r)$. Assume that no zeros of f lie on $\partial D(P,r)$. Then, counting the zeros of f according to multiplicity,

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \#\text{ zeros of } f \text{ inside } D(P,r). \qquad \text{§§5.1.5}$$

argument principle for meromorphic functions Let f be a holomorphic function on a domain $U \subseteq \mathbb{C}$. Assume that $\overline{D}(P,r) \subseteq U$, and that f has neither zeros nor poles on $\partial D(P,r)$. Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k,$$

where $n_1, n_2, \ldots, n_p$ are the multiplicities of the zeros $z_1, z_2, \ldots, z_p$ of f in $D(P,r)$ and $m_1, m_2, \ldots, m_q$ are the orders of the poles $w_1, w_2, \ldots, w_q$ of f in $D(P,r)$. §§5.1.7.

associative law If a, b, c are complex numbers, then

$$(a+b)+c = a+(b+c) \qquad \text{(Associativity of Addition)}$$

and

$$(a \cdot b) \cdot c = a \cdot (b \cdot c). \qquad \text{(Associativity of Multiplication)}$$

§§1.1.2, 1.1.6.

assumes the value β to order n A holomorphic function assumes the value β to order n at the point P if the function $f(z) - \beta$ vanishes to order n at P. §§5.1.3.

barrier Let $U \subseteq \mathbb{C}$ be an open set and $P \in \partial U$. We call a function $b : \overline{U} \to \mathbb{R}$ a *barrier* for U at P if

1. b is continuous;
2. b is subharmonic on U;
3. $b\big|_{\partial U} \leq 0$;
4. $\{z \in \partial U : b(z) = 0\} = \{P\}$.

§§7.7.9.

beta function If $\operatorname{Re} z > 0, \operatorname{Re} w > 0$, then the beta function of z and w is

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt.$$

§§13.1.11.

Bieberbach conjecture This is the problem of showing that each coefficient a_j of the power series expansion of a schlicht function satisfies $|a_j| \leq j$. In addition, the Köbe functions are the only ones for which equality holds. §§12.1.2.

biholomorphic mapping See *conformal mapping.*

Blaschke condition A sequence of complex numbers $\{a_j\} \subseteq D(0,1)$ satisfying

$$\sum_{j=1}^{\infty}(1-|a_j|) < \infty$$

is said to satisfy the Blaschke condition. §§9.1.5.

Blaschke factor This is a function of the form

$$B_a(z) = \frac{z-a}{1-\overline{a}z}$$

for some complex constant a of modulus less than one. See also *Möbius transformation.* §§9.1.1.

Blaschke factorization If f is a bounded holomorphic function or, more generally, a Hardy space function on the unit disc, then we may write

$$f(z) = z^m \cdot \left\{ \prod_{j=1}^{\infty} \frac{-\overline{a}_j}{|a_j|} B_{a_j}(z) \right\} \cdot F(z).$$

Here m is the order of the zero of f at $z = 0$, the points a_j are the other zeros of

f (counting multiplicities), the B_{a_j} are Blaschke factors, and F is a non-vanishing Hardy space function. §§9.1.7, §§12.3.7.

Blaschke product If $\{a_j\}$ satisfies the Blaschke condition, then the infinite product

$$\prod_{j=1}^{\infty} \frac{-\overline{a}_j}{|a_j|} B_{a_j}(z)$$

converges uniformly on compact subsets of the unit disc to define a holomorphic function B on $D(0,1)$. The function B is called a Blaschke product. §§9.1.6.

Bohr-Mollerup theorem Suppose that $\varphi : (0,\infty) \to (0,\infty)$ satisfies

1. $\log \varphi(x)$ is convex;
2. $\varphi(x+1) = x \cdot \varphi(x)$, all $x > 0$;
3. $\varphi(1) = 1$.

Then $\varphi(x) \equiv \Gamma(x)$, where Γ is the gamma function of Euler. Thus Γ is the only meromorphic function on $\mathbb{C}$ satisfying the functional equation $z\Gamma(z) = \Gamma(z+1), \Gamma(1) = 1$, and which is logarithmically convex on the positive real axis. §§13.1.10.

boundary maximum principle for harmonic functions Let $U \subseteq \mathbb{C}$ be a bounded domain. Let u be a continuous function on $\overline{U}$ that is harmonic on U. Then the maximum value of u on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must occur on ∂U. §§7.2.3.

boundary maximum principle for holomorphic functions Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Then the maximum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must occur on ∂U. §§5.4.2.

boundary minimum principle for harmonic functions Let $U \subseteq \mathbb{C}$ be a bounded domain. Let u be a continuous function on $\overline{U}$ that is harmonic on U. Then the minimum value of u on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must occur on ∂U. §§7.2.3.

boundary minimum principle for holomorphic functions Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Assume that f is non-vanishing. Then the minimum value of f on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must occur on ∂U. §§5.4.2.

boundary uniqueness for harmonic functions Let $U \subseteq \mathbb{C}$ be a bounded domain. Let u_1 and u_2 be continuous functions on $\overline{U}$ that are harmonic on U. If $u_1 = u_2$ on ∂U then $u_1 = u_2$ on all of $\overline{U}$. §§7.2.5.

bounded on compact sets Let $\mathcal{F}$ be a family of functions on an open set $U \subseteq \mathbb{C}$. We say that $\mathcal{F}$ is *bounded on compact sets* if for each compact set $K \subseteq U$, there is a constant $M = M_K$ such that for all $f \in \mathcal{F}$ and all $z \in K$ we have

$$|f(z)| \leq M. \qquad \text{§§8.4.3.}$$

bounded holomorphic function A holomorphic function f on a domain U is said to be bounded if there is a positive constant M such that

$$|f(z)| \leq M$$

for all $z \in U$. §§9.1.4.

Carathéodory's theorem Let $\varphi : \Omega_1 \to \Omega_2$ be a conformal mapping. If $\partial\Omega_1, \partial\Omega_2$ are Jordan curves (simple, closed curves), then φ (resp. φ^{-1}) extends one-to-one and continuously to $\partial\Omega_1$ (resp. $\partial\Omega_2$). §§12.2.2.

Casorati-Weierstrass theorem Let f be holomorphic on a deleted neighborhood of P and supposed that f has an essential singularity at P. Then the set of values of f is dense in the complex plane. §§4.1.6.

Cauchy estimates If f is holomorphic on a region containing the disc $D(P,r)$ and if $|f| \leq M$ on $D(P,r)$, then

$$\left|\frac{\partial^k}{\partial z^k} f(P)\right| \leq \frac{M \cdot k!}{r^k}. \qquad \text{§§3.1.2.}$$

Cauchy integral formula Let f be holomorphic on an open set U that contains the closed disc $\overline{D}(P,r)$. Let $\gamma(t) = P + re^{it}$. Then, for each $z \in D(P,r)$,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

See §§2.3.1. The formula is also true for certain more general curves (§§2.3.3).

Cauchy integral formula for an annulus Let f be holomorphic on an annulus $\{z \in \mathbb{C} : r < |z - P| < R\}$. Let $r < s < S < R$. Then for each $z \in D(P, S) \setminus \overline{D}(P, s)$ we have

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - P| = S} \frac{f(\zeta)}{\zeta - P}\, d\zeta - \frac{1}{2\pi i} \oint_{|\zeta - P| = s} \frac{f(\zeta)}{\zeta - P}\, d\zeta.$$

§§4.2.5.

Cauchy integral theorem If f is holomorphic on a disc U and if $\gamma : [a, b] \to U$ is a closed curve, then

$$\oint_\gamma f(z)\, dz = 0. \qquad \text{§§2.3.2.}$$

The formula is also true for certain more general curves (§§2.3.3).

Cauchy-Riemann equations If u and v are real-valued, continuously differentiable functions on the domain U, then u and v are said to satisfy the Cauchy-Riemann equations on U if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \qquad \text{§§1.3.2.}$$

Cauchy-Schwarz Inequality The statement that if $z_1, \ldots z_n$ and $w_1, \ldots, w_n$ are complex numbers, then

$$\left| \sum_{j=1}^{n} z_j w_j \right|^2 \leq \sum_{j=1}^{n} |z_j|^2 \sum_{j=1}^{n} |w_j|^2. \qquad \text{§§1.2.7.}$$

Cayley transform This is the function

$$f(z) = \frac{i - z}{i + z}$$

that conformally maps the upper half plane to the unit disc. §§6.3.5.

classification of singularities in terms of Laurent series Let the holomorphic function f have an isolated singularity at P, and let

$$\sum_{j=-\infty}^{\infty} a_j (z - P)^j$$

be its Laurent expansion. Then

- If $a_j = 0$ for all $j < 0$, then f has a removable singularity at P.
- If, for some $k < 0$, $a_j = 0$ for $j < k$ and $a_k \neq 0$, then f has a pole of order k at P.
- If there are infinitely many non-zero a_j with negative index j, then f has an essential singularity at P.

§§4.2.8.

clockwise The direction of traversal of a curve γ such that the region interior to the curve is always on the right. §§2.1.7.

closed curve A curve $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = \gamma(b)$. §§2.1.2.

closed disc of radius r and center P A disc in the plane having radius r and center P and *including* the boundary of the disc. §§1.1.5.

closed set A set E in the plane with the property that the complement of E is open. §§1.1.5.

commutative law If a, b, c are complex numbers, then

$$a + b = b + a \qquad \text{(Commutativity of Addition)}$$

and

$$a \cdot b = b \cdot a\,. \qquad \text{(Commutativity of Multiplication)}$$

§§1.1.6.

compact set A set $K \subseteq \mathbb{C}$ is compact if it is both closed and bounded. §§3.1.5.

complex derivative If f is a function on a domain U, then the complex derivative of f at a point P in U is the limit

$$\lim_{z \to P} \frac{f(z) - f(P)}{z - P}. \qquad \text{§§1.3.5.}$$

complex differentiable A function f is differentiable on a domain U if it possesses the complex derivative at each point of U. §§1.3.6.

complex line integral Let U be a domain, g a continuous function on U, and $\gamma : [a, b] \to U$ a curve. The complex line integral of g along γ is

$$\oint_\gamma g(z)\, dz \equiv \int_a^b g(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)\, dt. \qquad \text{§§2.1.6.}$$

complex numbers Any number of the form $x + iy$ with x and y real. §§1.1.2.

condition for the uniform convergence of an infinite product of functions Let $U \subseteq \mathbb{C}$ be a domain and let f_j be holomorphic functions on U. Assume that

$$\sum_{j=1}^{\infty} |f_j|$$

converges uniformly on compact subsets of U. Then the sequence of partial products

$$F_N(z) \equiv \prod_{j=1}^{N} (1 + f_j(z))$$

converges uniformly on compact sets to a holomorphic limit $F(z)$. We write

$$F(z) = \prod_{j=1}^{\infty} (1 + f_j(z)). \qquad \text{§§8.1.12, §§8.1.15.}$$

condition for the convergence of an infinite product of numbers If

$$\sum_{j=1}^{\infty} |a_j| < \infty,$$

then both

$$\prod_{j=1}^{\infty} (1 + |a_j|)$$

and

$$\prod_{j=1}^{\infty} (1 + a_j)$$

converge. §§8.1.11.

conformal A function f on a domain U is conformal if it preserves angles and dilates equally in all directions. A holomorphic function is conformal, and conversely. §§1.3.6, §§2.2.5.

conformal mapping Let U, V be domains in $\mathbb{C}$. A function $f : U \to V$ that is holomorphic, one-to-one, and onto is called a conformal mapping or conformal map. §§6.1.1.

conformal self-map Let $U \subseteq \mathbb{C}$ be a domain. A function $f : U \to U$ that is holomorphic, one-to-one, and onto is called a conformal (or biholomorphic) self-map of U. §§5.5.1.

conjugate If $z = x + iy$ is a complex number, then $\overline{z} = x - iy$ is its (complex) conjugate. §§1.1.2.

connected A set E in the plane is connected if there do not exist disjoint and non-empty open sets U and V such that $E = (U \cap E) \cup (V \cap E)$. §§1.3.1.

continuing a function element Finding additional function elements that are analytic continuations of the given function element. §§10.1.2, §§10.1.3, §§10.1.4.

continuous A function f with domain S is continuous at a point P in S if the limit of $f(x)$ as x approaches P is $f(P)$. An equivalent definition, coming from topology, is that f is continuous provided that, whenever V is an open set in the range of f, then $f^{-1}(V)$ is open in the domain of f. §§1.3.1, §§5.2.1.

continuously differentiable A function f with domain (the open set) S is continuously differentiable if the first derivative(s) of f exist at every point of S and if each of those first derivative functions is continuous on S. §§1.3.1, §§2.1.3.

continuously differentiable, k times A function f with domain S such that all derivatives of f up to and including order k exist and each of those derivative functions is continuous on S. §§1.3.1.

convergence of a Laurent series The Laurent series

$$\sum_{j=-\infty}^{\infty} a_j (z - P)^j$$

is said to converge if each of the power series

$$\sum_{j=-\infty}^{0} a_j(z-P)^j \qquad \text{and} \qquad \sum_{1}^{\infty} a_j(z-P)^j$$

converges. §§4.2.2.

convergence of an infinite product An infinite product

$$\prod_{j=1}^{\infty}(1+a_j)$$

is said to *converge* if

1. Only a finite number $a_{j_1}, \ldots, a_{j_k}$ of the a_j's are equal to -1.
2. If $N_0 > 0$ is so large that $a_j \neq -1$ for $j > N_0$, then

$$\lim_{N\to+\infty} \prod_{j=N_0+1}^{N} (1+a_j)$$

exists and is non-zero.

§§8.1.9.

convergence of a power series The power series

$$\sum_{j=0}^{\infty} a_j(z-P)^j$$

is said to converge at z if the partial sums $S_N(z)$ converge as a sequence of numbers. §§3.1.6.

converges uniformly See *uniform convergence.*

countable set A set S is countable if there is a one-to-one, onto function $f : S \to \mathbb{N}$. §§6.5.3

countably infinite set See *countable set.*

counterclockwise The direction of traversal of a curve γ such that the region interior to the curve is always on the left. §§2.1.7.

counting function This is a function from classical number theory that aids in counting the prime numbers. §§13.3.1.

curve A continuous function $\gamma : [a, b] \to \mathbb{C}$. §§2.1.1.

deformability Let U be a domain. Let $\gamma : [a, b] \to U$ and $\mu : [a, b] \to U$ be curves in U. We say that γ is deformable to μ in U if there is a continuous function $H(s, t)$, $0 \leq s, t \leq 1$ such that $H(0, t) = \gamma(t)$, $H(1, t) = \mu(t)$, and $H(s, t) \in U$ for all (s, t). §§2.3.4, §§10.3.2.

deleted neighborhood Let $P \in \mathbb{C}$. A set of the form $D(P, r) \setminus \{P\}$ is called a deleted neighborhood of P. §§4.1.2.

denumerable set A set that is either finite or countably infinite.

derivative with respect to z If f is a continuously differentiable function on a domain U, then the derivative of f with respect to z on U is

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f. \qquad \text{§§1.3.3.}$$

derivative with respect to $\overline{z}$ If f is a continuously differentiable function on a domain U, then the derivative of f with respect to $\overline{z}$ on U is

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f. \qquad \text{§§1.3.3.}$$

differentiable See *complex differentiable.*

direct analytic continuation Let (f, U) and (g, V) be function elements. We say that (g, V) is a *direct analytic continuation* of (f, U) if $U \cap V \neq \emptyset$ and $f = g$ on $U \cap V$. §§10.1.4.

Dirichlet problem on the disc Given a continuous function f on the boundary of the unit disc $\partial D(0, 1)$, find a continuous function u on $\overline{D}(0, 1)$ whose restriction to $\partial D(0, 1)$ equals f. §§7.3.4.

Dirichlet problem on a general domain Let $U \subseteq \mathbb{C}$ be a domain. Let f be a continuous function on ∂U. Find a continuous function u on $\overline{U}$ such that u agrees with f on ∂U. §§7.7.1, §§7.8.1, §§14.2.1.

disc of convergence A power series

$$\sum_{j=0}^{\infty} a_j (z-P)^j$$

converges on a disc $D(P,r)$, where

$$r = \frac{1}{\limsup_{j\to\infty} |a_j|^{1/j}}.$$

The disc $D(P,r)$ is the disc of convergence of the power series. §§3.1.6.

discrete set A set $S \subset \mathbb{C}$ is discrete if for each $s \in S$ there is an $\delta > 0$ such that $D(s,\delta) \cap S = \{s\}$. See also *isolated point.* §§3.2.2.

distributive law If a, b, c are complex numbers, then the distributive laws are

$$a \cdot (b+c) = ab + ac$$

and

$$(b+c) \cdot a = ba + ca.$$

§§1.1.6.

domain A set U in the plane that is both open and connected. §§1.3.1.

domain of a function The domain of a function f is the set of numbers or points to which f can be applied.

entire function A holomorphic function whose domain is all of $\mathbb{C}$. §§3.1.3.

equivalence class If $\mathcal{R}$ is an equivalence relation on a set S, then the sets $E_s \equiv \{s' \in S : (s,s') \in \mathcal{R}\}$ are called equivalence classes. See [KRA3] for more on equivalence classes and equivalence relations. §§10.1.6.

equivalence relation Let $\mathcal{R}$ be a relation on a set S. We call $\mathcal{R}$ an equivalence relation if $\mathcal{R}$ is

- **reflexive:** For each $s \in S$, $(s,s) \in \mathcal{R}$.
- **symmetric:** If $s, s' \in S$ and $(s,s') \in \mathcal{R}$, then $(s',s) \in \mathcal{R}$.
- **transitive:** If $(s,s') \in \mathcal{R}$ and $(s',s'') \in \mathcal{R}$, then $(s,s'') \in \mathcal{R}$.

An equivalence relation results in the set S being partitioned into *equivalence classes.* §§10.1.6

essential singularity If the point P is a singularity of the holomorphic function f, and if P is neither a removable singularity nor a pole, then P is called an essential singularity. §§4.1.4, §§4.1.6, §§4.2.8.

Euclidean algorithm The algorithm for long division in the theory of arithmetic. §§3.1.4.

Euler-Mascheroni constant The limit

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\log n\right\}$$

exists. The limit is a positive constant denoted by γ and is called the Euler-Mascheroni constant. §§13.1.7.

Euler product formula For $\operatorname{Re} z > 1$, the infinite product $\prod_{p\in P}(1-1/p^z)$ converges and

$$\frac{1}{\zeta(z)}=\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^z}\right). \qquad \text{§§13.2.2}$$

Here $\mathcal{P}=\{2,3,5,7,11,\ldots\}$ is the set of prime numbers.

exponential, complex The function e^z. §§1.2.1.

extended line The real line (lying in the complex plane) with the point at infinity adjoined. §§6.3.7.

extended plane The complex plane with the point at infinity adjoined. See *stereographic projection.* §§6.3.2.

extended real numbers The real numbers with the points $+\infty$ and $-\infty$ adjoined. §§4.2.3.

field A number system that is closed under addition, multiplication, and division by non-zero numbers and in which these operations are commutative and associative. §§1.1.6.

finite set A set S is finite if it can be put in one-to-one correspondence with a set of the form $\{1,2,\ldots,N\}$. §§4.4.1, §§4.6.7.

formula for the derivative Let $U \subseteq \mathbb{C}$ be an open set and let f be holomorphic on U. Then f is infinitely differentiable on U. Moreover, if $\overline{D}(P,r) \subseteq U$ and $z \in D(P,r)$, then

$$\left(\frac{\partial}{\partial z}\right)^k f(z) = \frac{k!}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta, \quad k = 0, 1, 2, \ldots.$$

§§3.1.1.

functional equation for the zeta function This is the relation

$$\zeta(1-z) = 2\zeta(z)\Gamma(z)\cos\left(\frac{\pi}{2}z\right) \cdot (2\pi)^{-z},$$

which holds for all $z \in \mathbb{C}$. §§13.2.7.

function element An ordered pair (f, U) where U is an open disc and f is a holomorphic function defined on U. §§10.1.3.

Fundamental Theorem of Algebra The statement that every non-constant polynomial has a root. §§1.1.7, §§3.1.4.

Fundamental Theorem of Calculus along Curves Let $U \subset \mathbb{C}$ be a domain and $\gamma = (\gamma_1, \gamma_2) : [a,b] \to U$ a C^1 curve. If $f \in C^1(U)$, then

$$f(\gamma(b)) - f(\gamma(b)) = \int_a^b \left(\frac{\partial f}{\partial x}(\gamma(t)) \cdot \frac{d\gamma_1}{dt} + \frac{\partial f}{\partial y}(\gamma(t)) \cdot \frac{d\gamma_2}{dt}\right) dt.$$

§§2.1.5.

gamma function If $\operatorname{Re} z > 0$, then define

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \qquad \text{§§13.1.1}$$

generalized circles and lines In the extended plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, a generalized line (generalized circle) is an ordinary line union the point at infinity. Topologically, an extended line is a circle. §§6.3.6.

genus of an entire function The maximum of the rank of f and of the degree of the polynomial g in the exponential in the Weierstrass factorization. §§9.3.5.

global analytic function We have an equivalence relation by way of analytic continuation on the set of function elements. The equivalence classes ([KRA3, p. 53]) induced by this relation are called global analytic functions. §§10.1.6, §§10.3.6.

greatest lower bound See *infimum.*

Hankel contour The contour of integration C_ϵ used in the definition of the Hankel function. §§13.2.4.

Hankel function The function

$$H_\epsilon(z) = \int_{C_\epsilon} u(w)\, dw,$$

where $C_\epsilon = C_\epsilon(\delta)$ is the Hankel contour. §§13.2.4.

Hardy space If $0 < p < \infty$, then we define $H^p(D)$ to be the class of those functions holomorphic on the disc and satisfying the growth condition

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

In this circumstance we write $f \in H^p(D)$. It is convenient to use the notation $\|f\|_{H^p}$ to denote the displayed quantity.

We also let $H^\infty(D)$ denote the class of bounded holomorphic functions on D, and we let $\|f\|_{H^\infty}$ denote the supremum of f on D. §§12.3.1.

harmonic A function u on a domain U is said to be harmonic of $\Delta u = 0$ on U, that is, if u satisfies the Laplace equation. §§1.4.1.

harmonic conjugate If u is a real-valued harmonic function on a domain U, then a real-valued harmonic function v on U is said to be conjugate to u if $h = u + iv$ is holomorphic. §§1.4.2.

Harnack inequality Let u be a non-negative, harmonic function $D(0, R)$. Then, for any $z \in D(0, R)$,

$$\frac{R - |z|}{R + |z|} \cdot u(0) \leq u(z) \leq \frac{R + |z|}{R - |z|} \cdot u(0).$$

Let u be a non-negative, harmonic function on $D(P, R)$. Then, for any $z \in D(P, R)$,

$$\frac{R - |z - P|}{R + |z - P|} \cdot u(P) \leq u(z) \leq \frac{R + |z - P|}{R - |z - P|} \cdot u(P). \qquad \text{§§7.6.1}$$

Harnack principle Let $u_1 \leq u_2 \leq \ldots$ be harmonic functions on a connected open set $U \subseteq \mathbb{C}$. Then either $u_j \to \infty$ uniformly on compact sets or there is a harmonic function u on U such that $u_j \to u$ uniformly on compact sets. §§7.6.2.

holomorphic A continuously differentiable function on a domain U is holomorphic if it satisfies the Cauchy-Riemann equations on U or (equivalently) if $\partial f/\partial \overline{z} = 0$ on U. §§1.1.2, §§1.3.4, §§1.3.5.

holomorphic function on a Riemann surface A function F is holomorphic on the Riemann surface $\mathcal{R}$ if $F \circ \pi^{-1} : \pi(U) \to \mathbb{C}$ is holomorphic for each open set U in $\mathcal{R}$ with π one-to-one on U. Here π is a coordinate (chart) map on $\mathcal{R}$. §§10.4.4.

homeomorphic Two open sets U and V in $\mathbb{C}$ are *homeomorphic* if there is a one-to-one, onto, continuous function $f : U \to V$ with $f^{-1} : V \to U$ also continuous. §§6.4.1.

homeomorphism A *homeomorphism* of two sets $A, B \subseteq \mathbb{C}$ is a one-to-one, onto continuous mapping $F : A \to B$ with a continuous inverse. §§6.4.1.

homotopic See *deformability, homotopy.* §§10.3.2.

homotopy Let W be a domain in $\mathbb{C}$. Let $\gamma_0 : [0,1] \to W$ and $\gamma_1 : [0,1] \to W$ be curves. Assume that $\gamma_0(0) = \gamma_1(0) = P$ and that $\gamma_0(1) = \gamma_1(1) = Q$. We say that γ_0 and γ_1 are *homotopic in* W (with fixed endpoints) if there is a continuous function

$$H : [0,1] \times [0,1] \to W$$

such that

1) $H(0,t) = \gamma_0(t)$ for all $t \in [0,1]$;
2) $H(1,t) = \gamma_1(t)$ for all $t \in [0,1]$;
3) $H(s,0) = P$ for all $s \in [0,1]$;
4) $H(s,1) = Q$ for all $s \in [0,1]$.

Then H is called a *homotopy* (with fixed endpoints) of the curve γ_0 to the curve γ_1. The two curves γ_0, γ_1 are said to be *homotopic.* §§10.3.2.

Hurwitz's theorem Suppose that $U \subseteq \mathbb{C}$ is a domain and that $\{f_j\}$ is a sequence of nowhere-vanishing holomorphic functions on U. If the sequence $\{f_j\}$ converges uniformly on compact subsets of U to a (necessarily holomorphic) limit function f_0, then either f_0 is nowhere-vanishing or $f_0 \equiv 0$. §§5.3.4.

image of a function The set of values taken by the function.

imaginary part If $z = x + iy$ is a complex number, then its imaginary part is y. §§1.1.2.

imaginary part of a function f If $f = u + iv$ is a complex-valued function, with u and v real-valued functions, then v is its imaginary part. §§1.3.2.

index Let U be a domain and $\gamma : [0,1] \to U$ a piecewise C^1 curve in U. Let $P \in U$ be a point that does not lie on γ. Then the index of γ with respect to P is defined to be

$$\mathrm{Ind}_\gamma(P) \equiv \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P}\, d\zeta.$$

The index is always an integer. §§4.4.4.

infimum Let $S \subseteq \mathbb{R}$ be a set of numbers. We say that a number m is an infimum for S if $m \leq s$ for all $s \in S$ and there is no number greater than m that has the same property. Every set of real numbers that is bounded below has an infimum. The term "greatest lower bound" has the same meaning.

infinite product An expression of the form $\prod_{j=1}^{\infty}(1 + a_j)$. §§8.1.6.

integer A whole number, or one of $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$.

integral representation of the beta function, alternate form For $z, w \notin \{0, -1, -2, \ldots\}$,

$$B(z,w) = 2\int_0^{\pi/2} (\sin\theta)^{2z-1}(\cos\theta)^{2w-1}\, d\theta.$$

§§13.1.14.

irrational numbers Those real numbers that have non-terminating, non-repeating decimal expansions. §§1.1.1.

isolated point A point s of a set $S \subseteq \mathbb{C}$ is said to be isolated if there is an $\delta > 0$ such that $D(s,\delta) \cap S = \{s\}$. §§3.2.2.

isolated singularity See *singularity.*

isolated singular point See *singularity.*

Jensen's formula Let f be holomorphic on a neighborhood of $\overline{D}(0,r)$ and suppose that $f(0) \neq 0$. Let $a_1, \ldots, a_k$ be the zeros of f in $D(0,r)$, counted according to their multiplicities. Assume that f does not vanish on $\partial D(0,r)$. Then

$$\log|f(0)| + \sum_{j=1}^{k} \log\left|\frac{r}{a_j}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta.$$

§§9.1.2.

Jensen's inequality Let f as in Jensen's formula. Then

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta.$$

§§9.1.3.

Jordan curve See *simple, closed curve.*

Köbe function Let $0 \leq \theta < 2\pi$. The *Köbe function*

$$f_\theta(z) \equiv \frac{z}{(1+e^{i\theta}z)^2}$$

is a schlicht function that satisfies $|a_j| = j$ for all j. §§12.1.1.

Köbe 1/4 theorem If f is schlicht, then

$$f(D(0,1)) \supseteq D(0,1/4). \qquad \text{§§12.1.5}$$

k times continuously differentiable function A function f with domain S such that all derivatives of f up to and including order k exist and

each of those derivatives is continuous on S. §§1.3.1.

Lambda function Define the function

$$\Lambda : \{n \in \mathbb{Z} : n > 0\} \to \mathbb{R}$$

by the condition

$$\Lambda(m) = \begin{cases} \log p & \text{if } m = p^k,\ p \in \mathcal{P},\ 0 < k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

(Here $\mathcal{P}$ is the collection of prime numbers.) §§13.2.10.

Laplace equation The partial differential equation

$$\triangle u = 0. \qquad \text{§§7.1.1}$$

Laplace operator or Laplacian This is the partial differential operator

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}. \qquad \text{§§7.1.2}$$

Laurent series A series of the form

$$\sum_{j=-\infty}^{\infty} a_j (z - P)^j.$$

See also *power series.* §§4.2.1.

Laurent series expansion about ∞ Fix a positive number R. Let f be holomorphic on a set of the form $\{z \in \mathbb{C} : |z| > R\}$. Define $G(z) = f(1/z)$ for $|z| < 1/R$. If the Laurent series expansion of G about 0 is

$$\sum_{j=-\infty}^{\infty} a_j z^j,$$

then the Laurent series expansion of f about ∞ is

$$\sum_{j=-\infty}^{\infty} a_j z^{-j}. \qquad \text{§§4.6.7}$$

least upper bound See *supremum.*

limit of the function f at the point P Let f be a function on a domain U. The complex number ℓ is the limit of the f at P if for each $\epsilon > 0$ there is a $\delta > 0$ such that, whenever $z \in U$ and $0 < |z - P| < \delta$, then $|f(z) - P| < \epsilon$. §§1.3.5.

linear fractional transformation A function of the form

$$z \mapsto \frac{az + b}{cz + d},$$

for a, b, c, d complex constants with $ac - bd \neq 0$. §§6.3.1.

Liouville's theorem If f is an entire function that is bounded, then f is constant. §§3.1.3.

locally A property is true locally if it is true on compact sets.

Lusin area integral Let $\Omega \subseteq \mathbb{C}$ be a domain and $\varphi : \Omega \to \mathbb{C}$ a one-to-one holomorphic function. Then $\varphi(\Omega)$ is a domain and

$$\text{area}(\varphi(\Omega)) = \int_{\Omega} |\varphi'(z)|^2 \, dxdy. \qquad \text{§§12.1.3}$$

maximum principle for harmonic functions If u is a harmonic function on a domain U and if P in U is a local maximum for u, then u is identically constant. §§7.2.1.

maximum principle for holomorphic functions If f is a holomorphic function on a domain U and if P in U is a local maximum for $|f|$, then f is identically constant. §§5.4.1.

maximum principle for subharmonic functions If u is subharmonic on U and if some point $P \in U$ is a local maximum for u, then u is identically constant. §§7.7.6.

mean value property for harmonic functions Let u be harmonic on an open set containing the closed disc $\overline{D}(P, r)$. Then

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta}) \, d\theta. \qquad \text{§§7.2.4}$$

This identity also holds for holomorphic functions.

Mergelyan's theorem Let $K \subseteq \mathbb{C}$ be compact and suppose that $\widehat{\mathbb{C}} \setminus K$ has only finitely many connected components. If $f \in C(K)$ is holomorphic on $\overset{\circ}{K}$ and if $\epsilon > 0$, then there is a rational function $r(z)$ with poles in $\widehat{\mathbb{C}} \setminus K$ such that

$$\max_{z \in K} |f(z) - r(z)| < \epsilon. \qquad \text{§§11.2.3}$$

Mergelyan's theorem for polynomials Let $K \subseteq \mathbb{C}$ be compact and assume that $\widehat{\mathbb{C}} \setminus K$ is connected. Let $f \in C(K)$ be holomorphic on $\overset{\circ}{K}$. Then for any $\epsilon > 0$ there is a holomorphic polynomial $p(z)$ such that

$$\max_{z \in K} |p(z) - f(z)| < \epsilon. \qquad \text{§§11.2.2}$$

meromorphic at ∞ Fix a positive number R. Let f be holomorphic on a set of the form $\{z \in \mathbb{C} : |z| > R\}$. Define $G(z) = f(1/z)$ for $|z| < 1/R$. We say that f is meromorphic at ∞ provided that G is meromorphic in the usual sense on $\{z \in \mathbb{C} : |z| < 1/R\}$. §§4.6.8.

meromorphic function Let U be a domain and $\{P_j\}$ a discrete set in U. If f is holomorphic on $U \setminus \{P_j\}$ and f has a pole at each of the $\{P_j\}$, then f is said to be meromorphic on U. §§4.6.1.

minimum principle for harmonic functions If u is a harmonic function on a domain U and if P in U is a local minimum for u, then u is identically constant. §§7.2.2.

minimum principle for holomorphic functions If f is a holomorphic function on a domain U, if f does not vanish on U, and if P in U is a local minimum for $|f|$, then f is identically constant. §§5.4.3.

Mittag-Leffler theorem Let $U \subseteq \mathbb{C}$ be any open set. Let $\alpha_1, \alpha_2, \ldots$ be a finite or countably infinite set of *distinct elements of* U with no accumulation point in U. Suppose, for each j, that U_j is a neighborhood of α_j. Further assume, for each j, that m_j is a meromorphic function defined on U_j with a pole at α_j and no other poles. Then there exists a meromorphic m on U such that $m - m_j$ is holomorphic on U_j for every j. §§8.3.6.

Mittag-Leffler theorem, alternative formulation Let $U \subseteq \mathbb{C}$ be any open set. Let $\alpha_1, \alpha_2, \ldots$ be a finite or countably infinite set of distinct

elements of U, having no accumulation point in U. Let s_j be a sequence of Laurent polynomials (or "principal parts"),

$$s_j(z) = \sum_{\ell=-p(j)}^{-1} a_\ell^j \cdot (z - \alpha_j)^\ell.$$

Then there is a meromorphic function on U whose principal part at each α_j is s_j. §§8.3.6.

Möbius transformation This is a function of the form

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

for a fixed complex constant a with modulus less than 1. Such a function φ_a is a conformal self-map of the unit disc. §§5.5.1.

modulus If $z = x + iy$ is a complex number, then $|z| = \sqrt{x^2 + y^2}$ is its modulus. §§1.1.4.

monodromy theorem Let $W \subseteq \mathbb{C}$ be a domain. Let (f, U) be a function element, with $U \subseteq W$. Let P denote the center of the disc U. Assume that (f, U) admits unrestricted continuation in W. If γ_0, γ_1 are each curves that begin at P, terminate at some point Q, and are homotopic in W, then the analytic continuation of (f, U) to Q along γ_0 equals the analytic continuation of (f, U) to Q along γ_1. §§10.3.5.

monogenic See *holomorphic.*

monotonocity of the Hardy space norm Let f be holomorphic on D. If $0 < r_1 < r_2 < 1$, then

$$\int_0^{2\pi} |f(r_1 e^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |f(r_2 e^{i\theta})|^p \, d\theta.$$

§§12.3.4.

Montel's theorem Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ be a family of holomorphic functions on an open set $U \subseteq \mathbb{C}$. If there is a constant $M > 0$ such that

$$|f(z)| \leq M \text{ , for all } z \in U \text{ , } f \in \mathcal{F},$$

then there is a sequence $\{f_j\} \subseteq \mathcal{F}$ such that f_j converges normally on U to a limit (holomorphic) function f_0. §§8.4.2.

Montel's theorem, second version Let $U \subseteq \mathbb{C}$ be an open set and let $\mathcal{F}$ be a family of holomorphic functions on U that is bounded on compact sets. Then there is a sequence $\{f_j\} \subseteq \mathcal{F}$ that converges normally on U to a limit (necessarily holomorphic) function f_0. §§8.4.4.

Morera's theorem Let f be a continuous function on a connected open set $U \subseteq \mathbb{C}$. If

$$\oint_\gamma f(z)\,dz = 0$$

for every simple, closed curve γ in U, then f is holomorphic on U. The result is true if it is only assumed that the integral is zero when γ is a rectangle, or when γ is a triangle. §§2.3.2.

multiple root Let f be either a polynomial or a holomorphic function on an open set U. Let k be a positive integer. If $P \in U$ and $f(P) = 0, f'(P) = 0, \ldots, f^{(k-1)}(P) = 0$, then f is said to have a multiple root at P. The root is said to be of order k. See *vanishes to order k*. §§3.1.4.

multiple singularities Let $U \subseteq \mathbb{C}$ be a domain and $P_1, P_2, \ldots$ be a discrete set in U. If f is holomorphic on $U \setminus \{P_j\}$ and has a singularity at each P_j, then f is said to have multiple singularities in U. §§4.4.1.

multiplicity of a zero or root The number k in the definition of *multiple root*. §§5.1.2.

neighborhood of a point in a Riemann surface We define neighborhoods of a "point" (f, U) in $\mathcal{R}$ by

$$\{(f_p, U_p) : p \in U \text{ and } (f_p, U_p) \text{ is a direct analytic continuation of } (f, U) \text{ to } p\}.$$

§§10.4.2.

normal convergence of a sequence A sequence of functions g_j on a domain U is said to converge normally to a limit function g if the f_j converge uniformly on compact subsets of U to g. §§8.1.3.

normal convergence of a series A series of functions $\sum_{j=1}^{\infty} g_j$ on a domain U is said to converge normally to a limit function g if the partial sums $S_N = \sum_{j=1}^{N} g_j$ converge uniformly on compact subsets of U to g. §§8.1.4

normal family Let $\mathcal{F}$ be a family of (holomorphic) functions with common domain U. We say that $\mathcal{F}$ is a normal family if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U, i.e., *converges normally* on U. See *Montel's theorem.* §§8.4.3.

one-to-one A function $f : S \to T$ is said to be one-to-one if whenever $s_1 \neq s_2$, then $f(s_1) \neq f(s_2)$. §§5.2.1.

onto A function $f : S \to T$ is said to be onto if whenever $t \in T$, then there is an $s \in S$ such that $f(s) = t$. §§5.2.1.

open disc of radius r and center P A disc $D(P, r)$ in the complex plane having radius r and center P and *not* including the boundary of the disc. §§1.1.5.

open mapping A function $f : S \to T$ is said to be open if whenever $U \subseteq S$ is open, then $f(U) \subseteq T$ is open. §§5.2.1.

open mapping theorem If $f : U \to \mathbb{C}$ is a holomorphic function on a domain U, then $f(U)$ will also be open. §§5.2.1.

open set A set U in the plane with the property that each point $P \in U$ has a disc $D(P, r)$ such that $D(P, r) \subseteq U$. §§1.1.5.

order of an entire function An entire function f is said to be of *finite order* if there exist numbers $a, r > 0$ such that

$$|f(z)| \leq \exp(|z|^a) \qquad \text{for all } |z| > r.$$

The infimum of all numbers a for which such an inequality holds is called the *order* of f and is denoted by $\lambda = \lambda(f)$. §§9.3.2

order of a pole See *pole.*

order of a root See *multiplicity of a root* and *vanishes to order k.*

Ostrowski-Hadamard gap theorem Let $0 < p_1 < p_2 < \cdots$ be integers and suppose that there is a $\lambda > 1$ such that

$$\frac{p_{j+1}}{p_j} > \lambda \quad \text{for} \qquad j = 1, 2, \ldots. \tag{9.2.1.1}$$

Suppose that, for some sequence of complex numbers $\{a_j\}$, the power series

$$f(z) = \sum_{j=1}^{\infty} a_j z^{p_j} \tag{9.2.1.2}$$

has radius of convergence 1. Then no point of ∂D is regular for f. §§9.2.2

partial fractions A method for decomposing a rational function into a sum of simpler rational components. Useful in integration theory, as well as in various algebraic contexts. See [THO] for details. §§15.3.2.

partial product For an infinite product $\prod_{j=1}^{\infty}(1+a_j)$, the partial product is

$$P_N = \prod_{j=1}^{N}(1 + a_j). \qquad \text{§§8.1.8}$$

partial sums of a power series If

$$\sum_{j=0}^{\infty} a_j(z - P)^j$$

is a power series, then its partial sums are the finite sums

$$S_N(z) \equiv \sum_{j=0}^{N} a_j(z - P)^j$$

for $N = 0, 1, 2, \ldots$. §§3.1.6.

path See curve. §§2.1.1.

path-connected Let $E \subseteq \mathbb{C}$ be a set. If, for any two points A and B in E there is a curve $\gamma : [0,1] \to E$ such that $\gamma(0) = A$ and $\gamma(1) = B$, then we say that E is path-connected. §§1.1.5.

Picard's Great Theorem Let U be a region in the plane, $P \in U$, and suppose that f is holomorphic on $U \setminus \{P\}$ and has an essential singularity at P. If $\epsilon > 0$, then the restriction of f to $U \cap [D(P, \epsilon) \setminus \{P\}]$ assumes all complex values except possibly one. §§10.5.3

Picard's Little Theorem If the range of an entire function f omits two points of $\mathbb{C}$, then f is constant. In other words, an entire function assumes

all complex values except possibly one. §§10.5.2.

piecewise C^k A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be piecewise C^k if

$$[a, b] = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{m-1}, a_m]$$

with $a = a_0 < a_1 < \cdots a_m = b$ and $\gamma\big|_{[a_{j-1}, a_j]}$ is C^k for $1 \leq j \leq m$. §§2.3.3.

π function For $x > 0$, this is the function

$$\pi(x) = \text{the number of prime numbers not exceeding } x.$$

§§13.3.2.

point at ∞ A point which is adjoined to the complex plane to make it topologically a sphere. §§6.3.1, §§6.3.3.

Poisson integral formula Let $u : U \to \mathbb{R}$ be a harmonic function on a neighborhood of $\overline{D}(0, 1)$. Then, for any point $a \in D(0, 1)$,

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\psi}) \cdot \frac{1 - |a|^2}{|a - e^{i\psi}|^2} \, d\psi. \qquad \text{§§7.3.1}$$

Poisson kernel for the unit disc This is the function

$$\frac{1}{2\pi} \frac{1 - |a|^2}{|a - e^{i\psi}|^2}$$

that occurs in the Poisson integral formula. §§7.3.2.

polar form of a complex number A complex number z written in the form $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in \mathbb{R}$. The number r is the modulus of z and θ is its argument. §§1.2.4.

polar representation of a complex number See *polar form*. §§1.2.4.

pole Let P be an isolated singularity of the holomorphic function f. If P is not a removable singularity for f but there exists a $k > 0$ such that $(z - P)^k \cdot f$ is a removable singularity, then P is called a pole of f. The least k for which this condition holds is called the order of the pole. §§4.1.4, §§4.2.8.

polynomial A polynomial is a function $p(z)$ [resp. $p(x)$] of the form

$$p(z) = a_0 + a_1 z + \cdots a_{k-1} z^{k-1} + a_k z^k,$$

[resp. $p(x) = a_0 + a_1 x + \cdots a_{k-1} x^{k-1} + a_k x^k$] where $a_0, \ldots, a_k$ are complex constants. §§1.1.7.

power series A series of the form

$$\sum_{j=0}^{\infty} a_j (z - P)^j.$$

More generally, the series can have any limits on the indices:

$$\sum_{j=m}^{\infty} a_j (z - P)^j \qquad \text{or} \qquad \sum_{j=m}^{n} a_j (z - P)^j. \qquad \text{§§3.1.6}$$

pre-vertices The inverse images of the corners of the polygon under study with the Schwarz-Christoffel mapping. §§14.4.1.

prime number This is an integer (whole number) that has no integer divisors except 1 and itself. The first few positive prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23$. By convention, 1 is not prime. §13.3.

prime number theorem This is the statement that

$$\lim_{x \to \infty} \frac{\pi(x)}{(x/\log x)} = 1.$$

Here $\pi(x)$ is the "prime number counting function." §§13.3.3

principal branch Usually that branch of a holomorphic function that focuses on values of the argument $0 \leq \theta < 2\pi$. The precise definition of "principal branch" depends on the particular function being studied. §§10.1.2, §§10.1.6.

principle of persistence of functional relations If two holomorphic functions defined in a domain containing the real axis agree for real values of the argument, then they agree at all points. §§3.2.3.

principal part Let f have a pole of order k at P. The negative power part

$$\sum_{j=-k}^{-1} a_j (z - P)^j$$

of the Laurent series of f about P is called the principal part of f at P. §§4.3.1.

range of a function Any set containing the image of the function.

rank of an entire function If f is an entire function and $\{a_j\}$ its zeros counting multiplicity, then the rank of f is the least positive integer p such that

$$\sum_{a_n \neq 0} |a_n|^{-(p+1)} < \infty.$$

We denote the rank of f by $p = p(f)$. §§9.3.5.

rational function A *rational* function is a quotient of polynomials. §§11.1.1.

rational number system Those numbers that are quotients of integers or whole numbers. A rational number has either terminating or repeating decimal expansions. §§1.1.1.

real analytic A function f of one or several real variables is called *real analytic* if it can locally be expressed as a convergent power series.

real number system Those numbers consisting of either terminating or non-terminating decimal expansions. §§1.1.1.

real part If $z = x+iy$ is a complex number, then its real part is x. §§1.1.2.

real part of a function f If $f = u + iv$ is a complex-valued function, with u, v real-valued functions, then u is its real part. §§1.3.2.

recursive identity for the gamma function If $\operatorname{Re} z > 0$, then

$$\Gamma(z+1) = z \cdot \Gamma(z). \qquad \text{§§13.1.2}$$

region See *domain*. §§1.3.1.

regular See *holomorphic*.

regular boundary point Let f be holomorphic on a domain U. A point P of ∂U is called *regular* if f extends to be a holomorphic function on an

open set containing U and also the point P. §§9.2.2.

relation Let S be a set. A relation on S is a collection of some (but not necessarily all) of the ordered pairs (s, s') of elements of S. See also *equivalence relation.* §§10.1.6

removable singularity Let P be an isolated singularity of the holomorphic function f. If f can be defined at P so as to be holomorphic in a neighborhood of P, then P is called a removable singularity for f . §§4.1.4, §§4.2.8.

residue If f has Laurent series

$$\sum_{j=-\infty}^{\infty} a_j (z-P)^j$$

about P, then the number a_{-1} is called the residue of f at P. We denote the residue by $\text{Res}_f(P)$. §§4.4.3.

residue, formula for Let f have a pole of order k at P. Then the residue of f at P is given by

$$\text{Res}_f(P) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}\left((z-P)^k f(z)\right)\Big|_{z=P}. \qquad \text{§§4.4.6}$$

residue theorem Let U be a domain and let the holomorphic function f have isolated singularities at $P_1, P_2, \ldots, P_m \in U$. Let $\text{Res}_f(P_j)$ be the residue of f at P_j. Also let $\gamma : [0,1] \to U \setminus \{P_1, P_2, \ldots, P_m\}$ be a piecewise C^1 curve. Let $\text{Ind}_\gamma(P_j)$ be the winding number of γ about P_j. Then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{m} \text{Res}_f(P_j) \cdot \text{Ind}_\gamma(P_j). \qquad \text{§§4.4.2}$$

Riemann hypothesis The celebrated Riemann Hypothesis is the conjecture that all the zeros of the zeta function ζ in the critical strip $\{z \in \mathbb{C} : 0 < \text{Re}\, z < 1\}$ actually lie on the critical line $\{z : \text{Re}\, z = 1/2\}$. §§13.2.9.

Riemann mapping theorem Let $U \subseteq \mathbb{C}$ be a simply connected domain, and assume that $U \neq \mathbb{C}$. Then there is a conformal mapping $\varphi : U \to D(0,1)$. §§6.4.3.

Riemann removable singularities theorem If P is an isolated singularity of the holomorphic function f and if f is bounded in a deleted neighborhood of P, then f has a removable singularity at P. §§4.1.5.

Riemann sphere See *extended plane.*

Riemann surface The idea of a Riemann surface is that one can visualize geometrically the behavior of function elements and their analytic continuations. A global analytic function is the set of all function elements obtained by analytic continuation along curves (from a base point $P \in \mathbb{C}$) of a function element (f, U) at P. Such a set, which amounts to a collection of convergent power series at different points of the plane $\mathbb{C}$, can be given the structure of a surface, in the intuitive sense of that word. §§10.4.1.

right turn angle The oriented angle of turning when traversing the boundary of a polygon that is under study with the Schwarz-Christoffel mapping. §§14.4.1

ring A number system that is closed under addition and multiplication. See also *field.* §§8.3.5.

root of a function or polynomial A value in the domain at which the function or polynomial vanishes. See also *zero.* §§1.1.7.

rotation A function $z \mapsto e^{ia}z$ for some fixed real number a. We sometimes say that the function represents "rotation through an angle a." §§6.2.1.

Rouché's theorem Let f, g be holomorphic functions on a domain $U \subseteq \mathbb{C}$. Suppose that $\overline{D}(P,r) \subseteq U$ and that, for each $\zeta \in \partial D(P,r)$,

$$|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|. \qquad (*)$$

Then the number of zeros of f inside $D(P,r)$ equals the number of zeros of g inside $D(P,r)$. The hypothesis $(*)$ is sometimes replaced in practice with

$$|f(\zeta) - g(\zeta)| < |g(\zeta)|$$

for $\zeta \in \partial D(P,r)$. §§5.3.1.

Runge's theorem Let $K \subseteq \mathbb{C}$ be compact. Let f be holomorphic on a *neighborhood* of K. Let $\mathcal{S} \subseteq \widehat{\mathbb{C}} \setminus K$ contain one point from each connected

component of $\widehat{\mathbb{C}} \setminus K$. Then, for any $\epsilon > 0$, there is a rational function $r(z)$ with poles in $\mathcal{S}$ such that

$$\max_{z \in K} |f(z) - r(z)| < \epsilon. \qquad \S\S 11.1.2$$

Runge's theorem, corollary for polynomials Let $K \subseteq \mathbb{C}$ be compact and assume that $\widehat{\mathbb{C}} \setminus K$ is connected. Let f be holomorphic on a neighborhood of K. Then for any $\epsilon > 0$ there is a holomorphic polynomial $p(z)$ such that

$$\max_{K} |p(z) - f(z)| < \epsilon. \qquad \S\S 11.1.3$$

schlicht function A holomorphic function f on the unit disc D is called *schlicht* if

1. f is one-to-one.
2. $f(0) = 0$.
3. $f'(0) = 1$.

In this circumstance we write $f \in \mathcal{S}$. §§12.1.1.

Schwarz-Christoffel mapping A conformal mapping from the upper half plane to a polygon. §§14.4.1

Schwarz-Christoffel parameter problem The problem of determining the pre-vertices of a Schwarz-Christoffel mapping. §§14.4.1.

Schwarz lemma Let f be holomorphic on the unit disc. Assume that

1. $|f(z)| \leq 1$ for all z.
2. $f(0) = 0$.

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

If either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation: $f(z) \equiv \alpha z$ for some complex constant α of unit modulus. §§5.5.1.

Schwarz-Pick lemma Let f be holomorphic on the unit disc. Assume that

1. $|f(z)| \leq 1$ for all z.

2. $f(a) = b$ for some $a, b \in D(0,1)$.

Then

$$|f'(a)| \leq \frac{1-|b|^2}{1-|a|^2}.$$

Moreover, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then

$$\left|\frac{b_2 - b_1}{1 - \overline{b}_1 b_2}\right| \leq \left|\frac{a_2 - a_1}{1 - \overline{a}_1 a_2}\right|.$$

There is a "uniqueness" result in the Schwarz-Pick Lemma. If either

$$|f'(a)| = \frac{1-|b|^2}{1-|a|^2} \qquad \text{or} \qquad \left|\frac{b_2 - b_1}{1 - \overline{b}_1 b_2}\right| = \left|\frac{a_2 - a_1}{1 - \overline{a}_1 a_2}\right|,$$

then the function f is a conformal self-mapping (one-to-one, onto holomorphic function) of $D(0,1)$ to itself. §§5.5.2.

Schwarz reflection principle for harmonic functions Let V be a connected open set in $\mathbb{C}$. Suppose that $V \cap (\text{real axis}) = \{x \in \mathbb{R} : a < x < b\}$. Set $U = \{z \in V : \operatorname{Im} z > 0\}$. Assume that $v : U \to \mathbb{R}$ is harmonic and that, for each $\zeta \in V \cap (\text{real axis})$,

$$\lim_{U \ni z \to \zeta} v(z) = 0.$$

Set $\widetilde{U} = \{\overline{z} : z \in U\}$. Define

$$\widehat{v}(z) = \begin{cases} v(z) & \text{if } z \in U \\ 0 & \text{if } z \in V \cap (\text{real axis}) \\ -v(\overline{z}) & \text{if } z \in \widetilde{U}. \end{cases}$$

Then $\widehat{v}$ is harmonic on $U^* \equiv U \cup \widetilde{U} \cup \{x \in \mathbb{R} : a < x < b\}$. §§7.5.2.

Schwarz reflection principle for holomorphic functions Let V be a connected open set in $\mathbb{C}$ such that $V \cap (\text{the real axis}) = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$. Set $U = \{z \in V : \operatorname{Im} z > 0\}$. Suppose that $F : U \to \mathbb{C}$ is holomorphic and that

$$\lim_{U \ni z \to x} \operatorname{Im} F(z) = 0$$

for each $x \in \mathbb{R}$ with $a < x < b$. Define $\widetilde{U} = \{z \in \mathbb{C} : \overline{z} \in U\}$. Then there is a holomorphic function G on $U^* \equiv U \cup \widetilde{U} \cup \{x \in \mathbb{R} : a < x < b\}$ such that $G|_U = F$. In fact $\varphi(x) \equiv \lim_{U \ni z \to x} \operatorname{Re} F(z)$ exists for each $x = x + i0 \in (a, b)$ and

$$G(z) = \begin{cases} F(z) & \text{if } z \in U \\ \varphi(x) + i0 & \text{if } z \in \{x \in \mathbb{R} : a < x < b\} \\ \overline{F(\overline{z})} & \text{if } z \in \widetilde{U}. \end{cases} \qquad \text{§§7.5.2}$$

simple, closed curve A curve $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = \gamma(b)$ but the curve crosses itself nowhere else. §§2.1.2.1

simple root Let f be either a polynomial or a holomorphic function on an open set U. If $f(P) = 0$ but $f'(P) \neq 0$, then f is said to have a simple root at P. See also *multiple root.* §§3.1.4.

simply connected A domain U in the plane is simply connected if one of the following three equivalent conditions holds: it has no holes, or if its complement has only one connected component, or if each closed curve in U is homotopic to a point. §§1.4.2.

singularity Let f be a holomorphic function on $D(P, r) \setminus \{P\}$ (that is, on the disc minus its center). Then the point P is said to be a singularity of f. §§4.1.4.

singularity at ∞ Fix a positive number R. Let f be holomorphic on the set $\{z \in \mathbb{C} : |z| > R\}$. Define $G(z) = f(1/z)$ for $|z| < 1/R$. Then

- If G has a removable singularity at 0, then we say that f has a removable singularity at ∞.
- If G has a pole at 0, then we say that f has a pole at ∞.
- If G has an essential singularity at 0, then we say that f has an essential singularity at ∞.

§§4.6.6.

small circle mean value property A continuous function h on a domain $U \subseteq \mathbb{C}$ is said to have this property if, for each point $P \in U$, there is a number $\epsilon_P > 0$ such that $\overline{D}(P, \epsilon_P) \subseteq U$ and, for every $0 < \epsilon < \epsilon_P$,

$$h(P) = \frac{1}{2\pi} \int_0^{2\pi} h(P + \epsilon e^{i\theta})\, d\theta.$$

A function with the small circle mean value property on U must be harmonic on U. §§7.4.1.

smooth curve A curve $\gamma : [a, b] \to \mathbb{C}$ is smooth if γ is a C^k function (where k suits the problem at hand, and may be ∞) and γ' never vanishes. §§7.7.2.

smooth deformability Deformability in which the function $H(s,t)$ is smooth. See *deformability.*

solution of the Dirichlet problem on the disc Let f be a continuous function on $\partial D(0,1)$. Define

$$u(z) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\psi}) \cdot \frac{1-|z|^2}{|z-e^{i\psi}|^2}\, d\psi & \text{if } z \in D(0,1) \\ f(z) & \text{if } z \in \partial D(0,1). \end{cases}$$

Then u is continuous on $\overline{D}(0,1)$ and harmonic on $D(0,1)$. §§7.3.4.

special function These are particular functions that arise in theoretical physics, partial differential equations, and mathematical analysis. See *gamma function, beta function.* §§13.0.1.

stereographic projection A geometric method for mapping the plane to a sphere. §§6.3.3.

subharmonic Let $U \subseteq \mathbb{C}$ be an open set and f a real-valued continuous function on U. Suppose that for each $\overline{D}(P,r) \subseteq U$ and every real-valued harmonic function h defined on a neighborhood of $\overline{D}(P,r)$ that satisfies $f \leq h$ on $\partial D(P,r)$, it holds that $f \leq h$ on $D(P,r)$. Then f is said to be subharmonic on U. §§7.7.4.

sub-mean value property Let $f : U \to \mathbb{R}$ be continuous. Then f satisfies the sub-mean value property if, for each $\overline{D}(P,r) \subseteq U$,

$$f(P) \leq \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})d\theta. \qquad \text{§§7.7.5}$$

supremum Let $S \subseteq \mathbb{R}$ be a set of numbers. We say that a number M is a supremum for S if $s \leq M$ for all $s \in S$ and there is no number less than M that has the same property. Every set of real numbers that is bounded above has a supremum. The term "least upper bound" has the same meaning.

topology A mathematical structure specifying open and closed sets and a notion of convergence. §§1.1.5.

triangle inequality The statement that if z, w are complex numbers then

$$|z+w| \leq |z| + |w|. \qquad \text{§§1.2.7}$$

uniform convergence for a sequence Let f_j be a sequence of functions on a set $S \subseteq \mathbb{C}$. The f_j are said to converge uniformly to a function g on S if for each $\epsilon > 0$ there is a number $N > 0$ such that if $j > N$, then $|f_j(s) - g(s)| < \epsilon$ for all $s \in S$. In other words, $f_j(s)$ converges to $g(s)$ at the same rate at each point of S. §§3.1.5.

uniform convergence for a series The series

$$\sum_{j=1}^{\infty} f_j(z)$$

on a set $S \subseteq \mathbb{C}$ is said to converge uniformly to a limit function $F(z)$ if its sequence of partial sums converges uniformly to F. Equivalently, the series converges uniformly to F if for each $\epsilon > 0$ there is a number $N > 0$ such that if $J > N$, then

$$\left| \sum_{j=1}^{J} f_j(z) - F(z) \right| < \epsilon$$

for all $z \in S$.

uniform convergence on compact subsets for a sequence Let f_j be a sequence of functions on a set $S \subseteq \mathbb{C}$. The f_j are said to converge uniformly on compact subsets of S to a function g on S if, for each compact $K \subseteq S$ and for each $\epsilon > 0$, there is a $N > 0$ such that if $j > N$, then $|f_j(k) - g(k)| < \epsilon$ for all $k \in K$. In other words, $f_j(k)$ converges to $g(k)$ at the same rate at each point of K. §§8.1.1.

uniform convergence on compact subsets for a series The series

$$\sum_{j=1}^{\infty} f_j(z)$$

on a set $S \subseteq \mathbb{C}$ is said to be uniformly convergent on compact sets to a limit function $F(z)$ if, for each $\epsilon > 0$ and each compact $K \subseteq S$, there is an $N > 0$ such that if $J > N$, then

$$\left| \sum_{j=1}^{N} f(z) - F(z) \right| < \epsilon$$

for every $z \in K$. In other words, the series converges at the same rate at each point of K. §§8.1.4, §§8.1.5.

uniformly Cauchy for a sequence Let g_j be a sequence of functions on a set $S \subseteq \mathbb{C}$. The sequence is uniformly Cauchy if, for each $\epsilon > 0$, there is an $N > 0$ such that for all $j, k > N$ and all $z \in S$ we have $|g_j(z) - g_k(z)| < \epsilon$. §§8.1.2.

uniformly Cauchy for a series Let $\sum_{j=1}^{\infty} g_j$ be a series of functions on a set $S \subseteq \mathbb{C}$. The series is uniformly Cauchy if, for each $\epsilon > 0$, there is an $N > 0$ such that: for all $M \geq L > N$ and all $z \in S$ we have $|\sum_{j=L}^{M} g_j(z)| < \epsilon$. §§8.1.5.

uniformly Cauchy on compact subsets for a sequence Let g_j be a sequence of functions on a set $S \subseteq \mathbb{C}$. The sequence is uniformly Cauchy on compact subsets of S if, for each K compact in S and each $\epsilon > 0$, there is an $N > 0$ such that for all $\ell, m > N$ and all $k \in K$ we have $|g_\ell(k) - g_m(k)| < \epsilon$. §§8.1.2.

uniformly Cauchy on compact subsets for a series Let $\sum_{j=1}^{\infty} g_j$ be a series of functions on a set $S \subseteq \mathbb{C}$. The series is uniformly Cauchy on compact subsets if, for each compact set K in S and each $\epsilon > 0$, there is an $N > 0$ such that for all $M \geq L > N$ and all $k \in K$ we have $|\sum_{j=L}^{M} g_j(k)| < \epsilon$. §§8.1.5.

uniqueness of analytic continuation Let f and g be holomorphic functions on a domain U. If there is a disc $D(P, r) \subseteq U$ such that f and g agree on $D(P, r)$, then f and g agree on all of U. More generally, if f and g agree on a set with an accumulation point in U, then they agree at all points of U. §§3.2.3.

unrestricted continuation Let W be a domain and let (f, U) be a function element in W. We say (f, U) admits *unrestricted continuation* in W if there is an analytic continuation (f_t, U_t) of (f, U) along every curve γ that begins at P and lies in W. §§10.3.4.

value of an infinite product If $\prod_{j=1}^{\infty}(1 + a_j)$ converges, then we define its value to be

$$\left[\prod_{j=1}^{N_0}(1 + a_j)\right] \cdot \lim_{N \to +\infty} \prod_{N_0+1}^{N} (1 + a_j).$$

See *convergence of an infinite product.* §§8.1.10.

vanishes to order k A holomorphic function on a domain U vanishes to order $k \geq 1$ at $P \in U$ if $f^{(0)}(P) = 0, \ldots, f^{(k-1)}(P) = 0$, but $f^{(k)}(P) \neq 0$.

vanishing of an infinite product of functions The function f defined on a domain U by the infinite product

$$f(z) = \prod_{j=1}^{\infty} (1 + f_j(z))$$

vanishes at a point $z_0 \in U$ if and only if $f_j(z_0) = -1$ for some j. The multiplicity of the zero at z_0 is the sum of the multiplicities of the zeros of the functions $1 + f_j$ at z_0. §§8.1.13.

Weierstrass factor These are the functions

$$E_0(z) = 1 - z$$

and, for $1 \leq p \in \mathbb{Z}$,

$$E_p(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

Weierstrass factors are used in the factorization of entire functions. See *Weierstrass factorization theorem.* §§8.2.2

Weierstrass factorization theorem Let f be an entire function. Suppose that f vanishes to order m at 0, $m \geq 0$. Let $\{a_n\}$ be the other zeros of f, listed with multiplicities. Then there is an entire function g such that

$$f(z) = z^m \cdot e^{g(z)} \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right).$$

Here, for each j, E_j is a Weierstrass factor. §§8.2.4

Weierstrass (canonical) product Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of non-zero complex numbers with no accumulation point in the complex plane (note, however, that the a_j's need not be distinct). If $\{p_n\}$ are positive integers that satisfy

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty$$

for every $r > 0$, then the infinite product

$$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

(called a Weierstrass product) converges uniformly on compact subsets of $\mathbb{C}$ to an entire function F. The zeros of F are precisely the points $\{a_n\}$, counted with multiplicity. §§8.2.2.

Weierstrass theorem Let $U \subseteq \mathbb{C}$ be any open set. Let $a_1, a_2, \ldots$ be a finite or infinite sequence in U (possibly with repetitions) that has no accumulation point *in* U. Then there exists a holomorphic function f on U whose zero set is precisely $\{a_j\}$. §§8.3.2.

whole number See *integer.*

winding number See *index.*

zero If f is a polynomial or a holomorphic function on an open set U, then $P \in U$ is a zero of f if $f(P) = 0$. See *root of a function or polynomial.* §§3.2.1.

zero set If f is a polynomial or a holomorphic function on an open set U, then the zero set of f is $\mathcal{Z} = \{z \in U : f(z) = 0\}$. §§3.2.1.

zeta function For $\operatorname{Re} z > 1$, define

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z \log n}. \qquad \text{§§13.2.1}$$

List of Notation

Notation	Meaning	Subsection
$\mathbb{R}$	real number system	1.1.1
$\mathbb{R}^2$	Cartesian plane	1.1.1
$\mathbb{C}$	complex number system	1.1.2
z, w, ζ	complex numbers	1.1.2
$z = x + iy$	complex numbers	1.1.2
$w = u + iv$	complex numbers	1.1.2
$\zeta = \xi + i\eta$	complex numbers	1.1.2
$\operatorname{Re} z$	real part of z	1.1.2
$\operatorname{Im} z$	imaginary part of z	1.1.2
$\overline{z}$	conjugate of z	1.1.2
$\lvert z \rvert$	modulus of z	1.1.4
$D(P, r)$	open disc	1.1.5
$\overline{D}(P, r)$	closed disc	1.1.5
D	open unit disc	1.1.5
$\overline{D}$	closed unit disc	1.1.5
$A \setminus B$	complement of B in A	1.1.5
e^z	complex exponential	1.2.1
!	factorial	1.2.1
$\cos z$	$\frac{e^{iz}+e^{-iz}}{2}$	1.2.1
$\sin z$	$\frac{e^{iz}-e^{-iz}}{2i}$	1.2.1
$\arg z$	argument of z	1.2.6
C^k	k times continuously differentiable	1.3.1 2.1.3
$f = u + iv$	real and imaginary parts of f	1.3.2
$\operatorname{Re} f$	real part of the function f	1.3.2
$\operatorname{Im} f$	imaginary part of the function f	1.3.2
$\partial f / \partial z$	derivative with respect to z	1.3.3

List of Notation, Continued

Notation	Meaning	Subsection
$\partial f/\partial \overline{z}$	derivative with respect to $\overline{z}$	1.3.3
$\lim_{z\to P} f(z)$	limit of f at the point P	1.3.5
df/dz	complex derivative	1.3.5
$f'(z)$	complex derivative	1.3.5
$\triangle$	the Laplace operator or Laplacian	1.4.1
γ	a curve	2.1.1
$\gamma\big\vert_{[c,d]}$	restriction of γ to $[c,d]$	2.1.2
$\oint_\gamma g(z)\,dz$	complex line integral of g along γ	2.1.6
$S_N(z)$	partial sum of a power series	3.1.6
$\mathrm{Res}_f(P)$	residue of f at P	4.4.3
$\mathrm{Ind}_\gamma(P)$	index of γ with respect to P	4.4.4
$\widehat{\mathbb{C}}$	the extended complex plane	6.3.1
$\mathbb{C}\cup\{\infty\}$	the extended complex plane	6.3.1
$L\cup\{\infty\}$	generalized circle	6.3.6
$\mathbb{R}\cup\{\infty\}$	extended real line	6.3.7
$\prod_{j=1}^{\infty}(1+a_j)$	infinite product	8.11.6
$P_N=\prod_{j=1}^{N}(1+a_j)$	partial product	8.1.8
$E_0(z)$	Weierstrass factor	8.2.1
$E_p(z)$	Weierstrass factor	8.2.1
$B_a(z)$	Blaschke factor	9.1.1
$B(z)$	Blaschke product	9.1.6
$\lambda(f)$	order of f	9.3.2.
$p(f)$	rank of f	9.3.5
$\mu(f)$	genus of f	9.3.5
$H:[0,1]\times[0,1]\to W$	a homotopy in W	10.3.2

List of Notation, Continued

Notation	Meaning	Subsection
$C(K)$	the continuous functions on K	11.2.2
$\mathcal{S}$	the class of Schlicht functions	12.1.1
$H^p(D)$	Hardy space	12.3.1
$\| \ \|_{H^p}$	Hardy space norm	12.3.1
$\Gamma(z)$	the Gamma function	13.1.1
γ	Euler-Mascheroni constant	13.1.7
$B(z,w)$	the beta function	13.1.11
$\zeta(z)$	the zeta function	13.2.1
$\mathcal{P}$	the set of prime numbers	13.2.2
$\pi(x)$	the number of primes $\leq x$	13.3.2
θ_j	right-turn angle	14.4.1
$\widehat{f}(n)$	Fourier coefficient of f	15.1.1
$Sf(t)$	Fourier series of f	15.1.1
$S_N f(t)$	partial sum of Fourier series of f	15.1.1
$\widehat{f}(\xi)$	Fourier transform of f	15.2.1
$g^\vee$	inverse Fourier transform of g	15.2.1
$F(s)$	Laplace transform of f	15.3
$\mathcal{L}(f)$	Laplace transform of f	15.3
$A(z)$	z-transform of $\{a_j\}$	15.4

Table of Laplace Transforms

Function	Laplace Transform	Domain of Convergence
e^{at}	$\frac{1}{s-a}$	$\{s : \operatorname{Re} s > \operatorname{Re} a\}$
1	$\frac{1}{s}$	$\{s : \operatorname{Re} s > 0\}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	w real, $\{s : \operatorname{Re} s > 0\}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	w real, $\{s : \operatorname{Re} s > 0\}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$	w real, $\{s : \operatorname{Re} s > \lvert\omega\rvert\}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$	ω real, $\{s : \operatorname{Re} s > \lvert\omega\rvert\}$
$e^{-\lambda t} \cos \omega t$	$\frac{s+\lambda}{(s+\lambda)^2+\omega^2}$	ω, λ real, $\{s : \operatorname{Re} s > -\lambda\}$
$e^{-\lambda t} \sin \omega t$	$\frac{\omega}{(s+\lambda)^2+\omega^2}$	ω, λ real, $\{s : \operatorname{Re} s > -\lambda\}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$\{s : \operatorname{Re} s > \operatorname{Re} a\}$
t^n	$\frac{n!}{s^{n+1}}$	$\{s : \operatorname{Re} s > 0\}$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	ω real, $\{s : \operatorname{Re} s > 0\}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	ω real, $\{s : \operatorname{Re} s > 0\}$
$f'(t)$	$s\mathcal{L}f(s) - f(0)$	$\{s : \operatorname{Re} s > 0\}$
$f''(t)$	$s^2\mathcal{L}f(s) - sf(0) - f'(0)$	$\{s : \operatorname{Re} s > 0\}$
$tf(t)$	$-(\mathcal{L}f)'(s)$	$\{s : \operatorname{Re} s > 0\}$
$e^{at}f(t)$	$\mathcal{L}f(s-a)$	$\{s : \operatorname{Re} s > 0\}$

A Guide to the Literature

Complex analysis is an old subject, and the associated literature is large. Here we give the reader a representative sampling of some of the resources that are available. Of course no list of this kind can be complete.

Traditional Texts

- L. V. Ahlfors, *Complex Analysis*, 2nd ed., McGraw-Hill, New York, 1966.
- L. V. Ahlfors, *Conformal Invariants*, McGraw-Hill, 1973.
- C. Carathéodory, *Theory of Functions of a Complex Variable*, Chelsea, New York, 1954.
- H. P. Cartan, *Elementary Theory of Analytic Functions of One and Several Complex Variables*, Addison-Wesley, Reading, 1963.
- E. T. Copson, *An Introduction to the Theory of Functions of One Complex Variable*, The Clarendon Press, Oxford, 1972.
- R. Courant, *The Theory of Functions of a Complex Variable*, New York University, New York, 1948.
- P. Franklin, *Functions of Complex Variables*, Prentice-Hall, Englewood Cliffs, 1958.
- W. H. J. Fuchs, *Topics in the Theory of Functions of One Complex Variable*, Van Nostrand, Princeton, 1967.
- B. A. Fuks, *Functions of a Complex Variable and Some of Their Applications*, Addison-Wesley, Reading, 1961.
- G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, Providence, 1969.
- K. Knopp, *Theory of Functions*, Dover, New York, 1945-1947.
- Z. Nehari, *Introduction to Complex Analysis*, Allyn & Bacon, Boston, 1961.
- R. Nevanlinna, *Introduction to Complex Analysis*, Chelsea, New York, 1982.

- W. F. Osgood, *Functions of a Complex Variable*, G. E. Stechert, New York, 1942.
- G. Polya and G. Latta, *Complex Variables*, John Wiley & Sons, New York, 1974.
- J. Pierpont, *Functions of a Complex Variable*, Ginn & Co., Boston, 1914.
- S. Saks and A. Zygmund, *Analytic Functions*, Nakl. Polskiego Tow. Matematycznego, Warsaw, 1952.
- G. Sansone, *Lectures on the Theory of Functions of a Complex Variable*, P. Noordhoff, Groningen, 1960.
- V. I. Smirnov and N. A. Lebedev, *Functions of a Complex Variable; Constructive Theory*, MIT Press, Cambridge, 1968.

Modern Texts

- A. Beardon, *Complex Analysis: The Argument Principle in Analysis and Topology*, John Wiley & Sons, New York, 1979.
- C. Berenstein and R. Gay, *Complex Variables: An Introduction*, Springer-Verlag, New York, 1991.
- R. P. Boas, *An Invitation to Complex Analysis*, Random House, New York, 1987.
- R. Burckel, *Introduction to Classical Complex Analysis*, Academic Press, New York, 1979.
- J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer-Verlag, New York, 1978.
- J. Duncan, *The Elements of Complex Analysis*, John Wiley & Sons, New York, 1968.
- S. D. Fisher, *Complex Variables*, 2nd ed., Brooks/Cole, Pacific Grove, 1990.
- A. R. Forsyth, *Theory of Functions of a Complex Variable*, 3rd ed., Dover, New York, 1965.
- A. O. Gel'fond, *Residues and their Applications*, Mir Publishers, Moscow, 1971.
- R. E. Greene and S. G. Krantz, *Function Theory of One Complex Variable*, John Wiley and Sons, New York, 1997.

- M. Heins, *Complex Function Theory*, Academic Press, New York, 1968.
- E. Hille, *Analytic Function Theory*, 2nd ed., Chelsea, New York, 1973.
- W. Kaplan, *A First Course in Functions of a Complex Variable*, Addison-Wesley, Cambridge, 1953.
- S. G. Krantz, *Complex Analysis: The Geometric Viewpoint*, Mathematical Association of America, Washington, D.C., 1990.
- S. Lang, *Complex Analysis*, 3rd ed., Springer-Verlag, New York, 1993.
- N. Levinson and R. M. Redheffer, *Complex Variables*, Holden-Day, San Francisco, 1970.
- A. I. Markushevich, *Theory of Functions of a Complex Variable*, Prentice-Hall, Englewood Cliffs, 1965.
- Jerrold Marsden, *Basic Complex Analysis*, Freeman, San Francisco, 1973.
- G. Mikhailovich, *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, Providence, 1969.
- R. Narasimhan, *Complex Analysis in One Variable*, Birkhäuser, Boston, 1985.
- T. Needham, *Visual Complex Analysis*, Oxford University Press, New York, 1997.
- J. Noguchi, *Introduction to Complex Analysis*, American Mathematical Society, Providence, 1998.
- B. Palka, *An Introduction to Complex Function Theory*, Springer, New York, 1991.
- R. Remmert, *Theory of Complex Functions*, Springer-Verlag, New York, 1991.
- W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
- B. V. Shabat, *Introduction to Complex Analysis*, American Mathematical Society, Providence, 1992.
- M. R. Spiegel, *Schaum's Outline of the Theory and Problems of Complex Variables*, McGraw-Hill, New York, 1964.

Applied Texts

- M. Abramowitz and I. A. Segun, *Handbook of Mathematical Functions*, Dover, New York, 1965.
- J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 6th ed., McGraw-Hill, New York, 1996.
- G. F. Carrier, M. Crook, and C. E. Pearson, *Functions of a Complex Variable: Theory and Technique*, McGraw-Hill, New York, 1966.
- W. Derrick, *Complex Analysis and Applications*, 2nd ed., Wadsworth, Belmont, 1984.
- A. Erdelyi, *The Bateman Manuscript Project*, McGraw-Hill, New York, 1954.
- P. Henrici, *Applied and Computational Complex Analysis*, John Wiley & Sons, New York, 1974-1986.
- A. Kyrala, *Applied Functions of a Complex Variable*, John Wiley and Sons, 1972.
- W. R. Le Page, *Complex Variables and the Laplace Transform for Engineers*, McGraw-Hill, New York, 1961.
- E. B. Saff and E. D. Snider, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, 2nd ed., Prentice-Hall, Englewood Cliffs, 1993.
- D. Zwillinger, et al, *CRC Standard Mathematical Tables and Formulas*, CRC Press, Boca Raton, 1996.

References

[AHL] Ahlfors, L., *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1979.

[BCH] Brown, J. and Churchill, R., *Complex Variables and Applications*, 6th ed., McGraw-Hill, New York, 1996.

[CCP] Carrier, G., Crook, M., and Pearson, C., *Functions of a Complex Variable: Theory and Technique*, McGraw-Hill, New York, 1966.

[CHA] Char, B., et al, *Maple V: Library Reference Manual*, Springer-Verlag, New York, 1991.

[COH] Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Wiley-Interscience, New York, 1953.

[CRC] Zwillinger, D., *et al*, *CRC Standard Mathematical Tables and Formulae*, 30th ed., CRC Press, Boca Raton, 1996.

[DBR] De Branges, L., A Proof of the Bieberbach conjecture, *Acta Math.* 154(1985), 137-152.

[DUR] Duren, P., *Univalent Functions*, Springer Verlag, New York, 1983.

[DYM] Dym, H. and McKean, H., *Fourier Series and Integrals*, Academic Press, New York, 1972.

[EDW] Edwards, H., *Riemann's Zeta Function*, John Wiley and Sons, New York, 1974.

[FAK] Farkas, H. and Kra, I., *Riemann Surfaces*, 2nd ed., Springer, New York and Berlin, 1992.

[FOL] Folland, G. B., *Real Analysis: Modern Techniques and their Applications*, John Wiley and Sons, New York, 1984.

[GK] Greene, R. E. and Krantz, S. G., *Function Theory of One Complex Variable*, John Wiley and Sons, New York, 1997.

[HAL] Hanselman, D. and Littlefield, B., *The Student Edition of MATLAB*, Prentice-Hall, Upper Saddle River, 1997.

[HEN] Henrici, P., *Applied and Computational Complex Analysis*, Vol. I and II, John Wiley and Sons, New York, 1974, 1977.

[HER] Herstein, I., *Topics in Algebra*, Blaisdell, New York, 1964.

[HOR] Hörmander, L., *Notions of Convexity*, Birkhäuser, Boston, 1994.

[HUN] T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1980.

[KAT] Katznelson, Y., *An Introduction to Harmonic Analysis*, John Wiley and Sons, New York, 1968.

[KOB] Kober, H., *Dictionary of Conformal Representations*, 2nd ed., Dover Publications, New York, 1957.

[KRA1] Krantz, S., *Complex Analysis: The Geometric Viewpoint*, Mathematical Association of America, Washington, D.C., 1990.

[KRA2] Krantz, S., *Real Analysis and Foundations*, CRC Press, Boca Raton, 1992.

[KRA3] Krantz, S., *The Elements of Advanced Mathematics*, CRC Press, Boca Raton, 1995.

[KRA4] Krantz, S., *A Panorama of Harmonic Analysis*, Mathematical Association of America, Washington, D.C., 1999.

[KYT] Kythe, P., *Computational Conformal Mapping*, Birkhäuser, Boston, 1998.

[LOG] Logan, J. D., *Applied Mathematics*, 2nd ed., John Wiley and Sons, New York, 1997.

[LST] Loomis, L. and S. Sternberg, *Advanced Calculus*, Addison-Wesley, Reading, 1968.

[MAT] The Math Works, *MATLAB: The Language of Technical Computing*, The Math Works, Natick, Mass., 1997.

[MOC] Moler, C. and Costa, P., *MATLAB Symbolic Math Toolbox*, The Math Works, Inc., Natick, Mass., 1997.

[NEH] Nehari, Z., *Conformal Mapping*, Dover Publications, New York, 1952.

[ONE] B. O'Neill, *Elementary Differential Geometry*, 2nd ed., Academic Press, San Diego, 1997.

[RUD1] Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1953.

[RUD2] Rudin, W., *Real and Complex Analysis*, McGraw-Hill, New York, 1966.

[SASN] Saff, E. and Snider, A., *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, 2nd ed., Prentice-Hall, New York, 1993.

[STA] Stanley, R. P., *Combinatorics and Commutative Algebra*, Birkhäuser Publishing, Boston, 1983.

[STW] Stein, E. M. and Weiss, G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, 1971.

[THO] Thomas, E. and Finney, R., *Calculus and Analytic Geometry*, 6th ed., Addison-Wesley, Reading, MA, 1984.

[WOL] Wolfram, S., *The Mathematica Book*, 3rd ed., Cambridge University Press, Cambridge, 1996.

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